On Generating Functions of Two Variables

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Abstract. In this present paper, our objective is to introduce the generating functions of two variables and to obtain the differential recurrence relations.

Key words: Generating functions, Recurrence relations

Mathematics subject classification: 33E20, 33E30

1. INTRODUCTION

The theory of generating functions play a very important role in the study of Discrete Mathematics and theory of special functions, which has drawn the attention of several Mathematician and computer scientist time to time. McBride [1] defined generating function as follows:

Let \( G(x,t) \) be a function that can be expanded in formal powers of \( t \) such that

\[
G(x,t) = \sum_{n=0}^{\infty} (c)_n g_n(x) t^n
\]

(1.1)

where \( c_n \) is a function of \( n \) that may contain the parameters of the set \( g_n(x) \), but is independent of \( x \) and \( t \). \( G(x,t) \) is called a generating function of the set \( g_n(x) \).

A generating function may be used to define a set of functions and to determine a differential recurrence relation or a pure recurrence relation.

In 1960 Rainville [2] devoted a chapter to the generating function and its differential recurrence relations, Srivastava and Manocha [3] wrote a book on Generating functions, in which they discussed the varieties of generating functions and techniques including Lie algebraic technique.

In continuation to the study, here we define generating function of two variables as:
If \( G(x,y,t) \) be the function, which has formal power series expansion in \( t \) as:

\[
G(x,y,t) = \sum_{n=0}^{\infty} c_n g_n(x,y) t^n
\]

(1.2)

If \( c_n \) and \( g_n(x,y) \) are assigned, we can determine some function \( G(x,y,t) \) as a finite sum of the products of a finite number of known special functions of two variables.

2. Main Results

**THEOREM-1**

Let the generating function \( g_n(x,y) \) is a function of two variables defined as

\[
\sum_{n=0}^{\infty} g_n(x,y) t^n = e^{\Phi_1(xt)\Phi_2(yt)}
\]

(2.1)

It follows that \( g_0(x,y) = 0 \), and for \( n \geq 1 \)

\[
x g_{n_1}(x,y) + y g_{n_2}(x,y) - ng_n(x,y) + g_{n-1}(x,y) = 0.
\]

(2.2)

**Proof of Theorem-1**

From (2.1), the generating function \( g_n(x,y) \) of a polynomial having two variables is given by

\[
\sum_{n=0}^{\infty} g_n(x,y) t^n = e^{\Phi_1(xt)\Phi_2(yt)}
\]

Let \( F = e^{\Phi_1(xt)\Phi_2(yt)} \)

then,

\[
x \frac{\partial F}{\partial x} = tex^e\Phi_1'(xt)\Phi_2(yt) \quad , \quad y \frac{\partial F}{\partial y} = tye^\Phi_1(xt)\Phi_2'(yt)
\]

By using (2.3), we get

\[
x g_{n_1}(x,y) + y g_{n_2}(x,y) - ng_n(x,y) + g_{n-1}(x,y) = 0
\]

where \( g_n(x,y) = \frac{\partial}{\partial x} g_n(x,y) \) and \( g_n(y) = \frac{\partial}{\partial y} g_n(x,y) \).

**THEOREM-2**

From (2.1), we get \( e^{\Phi_1(xt)\Phi_2(yt)} = \sum_{n=0}^{\infty} g_n(x,y) t^n \)

It follows that for arbitrary \( c \) as

\[
(1-t)^{-\sum_{n=0}^{\infty} c_n g_n(x,y) t^n}
\]

(2.4)

in which \( F(u,v) = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (c)_{r+k} u^r v^k \left( \frac{xt}{1-t} \right)^k \left( \frac{yt}{1-t} \right)^v \).
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Proof of Theorem 2

Here, we consider that the function \( \Phi \) in (2.1) has the formal power series expansion

\[
\Phi_n = \sum_{k=0}^{\infty} u_k x^k y^r \text{ where } x = xt \quad \text{and} \quad y = yt
\]

(2.5, a,b)

by applying (2.5,a,b) in (2.1) it becomes

\[
\sum_{n=0}^{\infty} g_n (x, y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{r=0}^{n-k} \frac{u_k x^k y^r}{(n-k-r)!} t^n
\]

On equating the coefficient of \( t^n \) on both sides, we get

\[
g_n (x, y) = \sum_{k=0}^{n} \sum_{r=0}^{n-k} \frac{u_k x^k y^r}{(n-k-r)!}
\]

By taking the sum we get

\[
\sum_{n=0}^{\infty} (c)_n g_n (x, y) t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{r=0}^{n-k} \frac{(c)_n x^k y^r}{(n-k-r)!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{r=0}^{n-k} \frac{(c+k+r)_n t^n (c)_{k+r} x^k (yt)^r}{n!}
\]

\[
= \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \frac{(c)_{k+r} x^k (yt)^r}{(1-t)^{c+k+r}} = \frac{F(xt, yt)}{(1-t)^{c+k+r}}
\]

THEOREM 3

From \( e^{\phi_1 (xt) \phi_2 (yt)} = \sum_{n=0}^{\infty} g_n (x, y) t^n \)

It follows that \( g_0 (x, y) = 0, g_1 (x, y) = 0 \), and for \( n \geq 2 \)

\[
x^2 g_{n_0} (x, y) + 2xy g_{n_0} (x, y) + y^2 g_{n_0} (x, y) - n(n-1) g_n (x, y)
\]

\[
+2x g_{(n-1)} (x, y) + 2y g_{(n-1)} (x, y) + g_{n-2} (x, y) = 0
\]

Proof of Theorem 3.

Let \( F = e^{\phi_1 (xt) \phi_2 (yt)} \), it is very easy to verify the following result,

\[
\frac{\partial^2 F}{\partial t^2} = x^2 \frac{\partial^2 F}{\partial x^2} + 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} + 2xt \frac{\partial F}{\partial x} + 2yt \frac{\partial F}{\partial y} + t^2 F
\]

Consider \( F = \sum_{n=0}^{\infty} g_n (x, y) t^n \), it gives

\[
\sum_{n=0}^{\infty} \frac{n-1}{2} g_{n-1} (x, y) t^n = \frac{\sum_{n=0}^{\infty} g_n (x, y) t^n + 2 \sum_{n=0}^{\infty} g_n (x, y) t^n + \sum_{n=0}^{\infty} g_n (x, y) t^n + 2 \sum_{n=0}^{\infty} g_n (x, y) t^n + \sum_{n=0}^{\infty} g_n (x, y) t^n}{n-1} + \sum_{n=0}^{\infty} g_n (x, y) t^n
\]

The above equation can be easily reduces to

\[
x^2 g_{n_0} (x, y) + 2xy g_{n_0} (x, y) + y^2 g_{n_0} (x, y) - n(n-1) g_n (x, y) + 2x g_{(n-1)} (x, y) + 2y g_{(n-1)} (x, y) + g_{n-2} (x, y) = 0
\]
THEOREM 4
Let the generating function of the polynomial \( q_n(x, y) \) is defined by

\[
\sum_{n=0}^{\infty} q_n(x, y)t^n = A(t)\Phi_1(xt)\Phi_2(yt) \tag{2.6}
\]
which satisfying following conditions

\[
A(t) = \sum_{n=0}^{\infty} a_n t^n, a_0 \neq 0, \quad \Phi_1(t) = \sum_{n=0}^{\infty} \nu_t t^n, \nu_0 \neq 0 \quad \text{and} \quad \Phi_2(t) = \sum_{n=0}^{\infty} \sigma_n t^n, \sigma_0 \neq 0 \quad (2.7, \text{a, b, c})
\]
is a polynomial in \( x \) and \( y \) and \( q_n(x, y) \) is of degree precisely \( 2n \) if and only if \( \nu_0 \neq 0 \), \( \sigma_0 \neq 0 \).

Proof of Theorem 4.
From (2.6) we get \( \sum_{n=0}^{\infty} q_n(x, y)t^n = A(t)\Phi_1(xt)\Phi_2(yt) \), where

\[
q_n(x, y) = \sum_{r, k=0}^{\infty} S(n,k,r)x^r y^k \tag{2.8}
\]
By putting the value of (2.8) in (2.6), it yields

\[
A(t)\Phi_1(xt)\Phi_2(yt) = \sum_{n=0}^{\infty} S(n,r,k)x^r y^k t^n \tag{2.9}
\]
By differentiating (2.9) w.r.t. ‘\( x \)’ \( m \) times and afterwards putting \( x = 0 \), we get

\[
A(t)\Phi_1^m(0)\Phi_2(0)yt^m = \sum_{n=0}^{\infty} S(n,k,m)m! \cdot y^k t^n \tag{2.10}
\]
Further differentiating w.r.t. ‘\( y \)’ \( m \) times and putting \( y = 0 \), we get

\[
A(t)\Phi_1^m(0)\Phi_2^m(0)t^{2m} = \sum_{n=0}^{\infty} S(n,m)(m!)^2 t^n \tag{2.11}
\]
On applying (2.7, a, b, c) in (2.10) it becomes

\[
A(t)\Phi_1^m(0)\Phi_2^m(0)t^{2m} = a_0 \nu_m \sigma_m (m!)^2 t^{2m} + \sum_{n=2m+1}^{\infty} C(n,m)t^n \tag{2.12a, b}
\]
By comparing (2.10) and (2.11), this leads to

\[
S(n, m) = 0 \quad \text{for} \quad n < 2m \quad \text{and} \quad S(n, m) = a_0 \nu_m \sigma_m \quad \text{for} \quad n = 2m . \tag{2.12a, b}
\]
Hence, (2.12, a) and (2.12, b) shows that \( q_n(x, y) \) is a polynomial of degree precisely \( 2n \) if and only if \( \sigma_m \neq 0, \nu_m \neq 0 \) and \( a_0 \neq 0 \).

THEOREM 5
For the polynomial defined in (2.6), satisfying (2.7, a, b, c), where \( \nu_n \neq 0 \) and \( \sigma_n \neq 0 \) then there exist a sequence of numbers \( \alpha_k \) such that, for \( n \geq 1 \)

\[
x q_n(x, y) + y q_n(x, y) - n q_n(x, y) - \sum_{k=0}^{n-1} \alpha_k q_{n-1-k}(x, y) = 0 . \tag{2.13}
\]
Proof of Theorem - 5

Let \( \frac{tA(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} \) and \( F = A(t) \Phi_1(xt) \Phi_2(yt) \)  
(2.14)

Then \( \frac{\partial F}{\partial x} = A(t) \Phi_1(xt) \Phi_2(yt)t \), \( \frac{\partial F}{\partial y} = A(t) \Phi_1(xt) \Phi_2(yt)t \)  
(2.15, a, b)

\( \frac{\partial F}{\partial t} = A(t) \Phi_1(xt) \Phi_2(yt) + xA(t) \Phi_1(xt) \Phi_2(yt) + yA(t) \Phi_1(xt) \Phi_2(yt) \)  
(2.16)

Eliminating \( \Phi_1, \Phi_2, \Phi_1 \) and \( \Phi_2 \) from equations (2.14), (2.15, a, b) and afterwards substituting these values in equation (2.16), finally we arrive at

\[ \frac{\partial F}{\partial t} = \frac{tA(t)}{A(t)} F + x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} \]

as \( \frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} F + x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} \)

Since \( F = \sum_{n=0}^{\infty} q_n(x, y) t^n \), it becomes

\[
\sum_{n=0}^{\infty} nq_n(x, y) t^n = \left[ \sum_{n=0}^{\infty} \alpha_n t^{n+1} \right] \left[ \sum_{n=0}^{\infty} q_n(x, y) t^n \right] + x \sum_{n=0}^{\infty} q_n(x, y) t^n + y \sum_{n=0}^{\infty} q_n(x, y) t^n \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \alpha_k q_{n-k}(x, y) t^n + x \sum_{n=0}^{\infty} q_n(x, y) t^n + y \sum_{n=0}^{\infty} q_n(x, y) t^n \\
= \sum_{n=1}^{\infty} \sum_{k=0}^{n} \alpha_k q_{n-1-k}(x, y) t^n + x \sum_{n=0}^{\infty} q_n(x, y) t^n + y \sum_{n=0}^{\infty} q_n(x, y) t^n \\
\]

Hence, we can say that

\[ xq_{n_1}(x, y) + yq_{n_2}(x, y) - nq_{n}(x, y) - \sum_{k=0}^{n-1} \alpha_k q_{n-1-k}(x, y) = 0 \]

In particular, if we take \( \frac{A(t)}{A(t)} = 1 \), then \( q_n(x, y) \) is same as \( g_n(x, y) \) [as defined (2.1)], therefore (2.13) is equal to (2.2).

**THEOREM - 6**

Let \( q_n(x, y) \) is defined as \( \sum_{n=0}^{\infty} q_n(x, y) t^n = A(t) \psi(xH(t), yH(t)) \)  
(2.17)

where \( \psi(t_1, t_2) = \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \sigma_r t_1^r t_2^k \), \( \sigma_{0,0} \neq 0 \) and \( H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, h_0 \neq 0 \)  
(2.18, a, b)

and satisfying (2.7, a), then \( q_n(x, y) \) is a polynomial of two variables of the degree precisely 2n if and only if \( \sigma_r \neq 0, \psi h \neq 0 \)

**Proof of Theorem - 6**

From (2.17) \( \sum_{n=0}^{\infty} q_n(x, y) t^n = A(t) \psi(xH(t), yH(t)) \)
Taking \( q_n(x, y) = \sum_{r,k=0}^{\infty} \sum_{r,k=0}^{\infty} S(n,k,r)x^ry^k \)

Now substituting the value of \( q_n(x, y) \) in (2.17), we get

\[
A(t)\psi(xH(t), yH(t)) = \sum_{n,r,k=0}^{\infty} S(n,r,k)x^ry^kt^n
\]  

(2.19)

On differentiating (2.19) w.r.t. ‘x’ \( m \) times and by putting \( x = 0 \), we get

\[
A(t)\psi_u^m(0, yH(t))(H(t))^{m_0} = \sum_{n,r,k=0}^{\infty} S(n,m,m_0) m_0!x^m t^n, \text{ here } u = xH(t)
\]

Again differentiating w.r.t ‘y’ \( m \) times and putting \( y = 0 \), we get

\[
A(t)\psi_v^m(0, 0)(H(t))^{m_0} = \sum_{n,r,k=0}^{\infty} S(n,m,m_0) m_0!y^m t^n, \text{ here } v = yH(t)
\]

(2.20)

For \( m_1 = m_2 \), equation (2.20) and (2.21) becomes

\[
A(t)\psi_u^m(0, 0)(H(t))^{m_0} = \sum_{n,r,k=0}^{\infty} S(n,m,m_0) m_0!x^m t^n
\]

(2.22)

and

\[
A(t)\psi_v^m(0, 0)(H(t))^{m_0} = a_0\sigma_m\nu_m h_0^{m_0} + \sum_{n=r_0}^{\infty} C(n,m,m_0) t^n
\]

(2.23)

By comparing (2.22) and (2.23), we get

\[
S(m,n) = 0 \text{ For } n < 2m \quad \text{ and } \quad S(n,m) = a_0\sigma_m\nu_m h_0^{m_0}, \text{ For } n = 2m
\]

Hence, from aforesaid equations, we arrive at a conclusion that \( q_n(x, y) \) is a polynomial of degree precisely \( 2n \) if and only if \( \sigma_m \neq 0, \nu_m \neq 0 \) and \( a_0 h_0 \neq 0 \).

**THEOREM 7**

If the polynomial \( q_n(x, y) \) is defined by (2.17) and holding the conditions (2.7, a), (2.18, a) and (2.18, b), then there exist a sequence of numbers \( \alpha_k \) and \( \beta_k \) such that for \( n \geq 1 \)

\[
xq_n(x, y) + yq_n(x, y) - nq_n(x, y) + \sum_{k=0}^{n-1} \alpha_k q_{n-k}(x, y) + x \sum_{k=0}^{n-1} \beta_k q_{n-k}(x, y) + y \sum_{k=0}^{n-1} \beta_k q_{n-k}(x, y) = 0
\]

Where \( \sigma_r \neq 0, \nu_k \neq 0 \)

**Proof of Theorem- 7**

From (2.14), we have

\[
\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n t^{n+1}
\]

Let \( \frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \) and \( F = A(t)\psi(xH(t), yH(t)) \)

(2.24, a, b)

By setting \( u = xH(t) \) and \( v = yH(t) \), it gives
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\[ \frac{\partial F}{\partial x} = A(t) \frac{\partial \psi}{\partial u} H(t) , \quad \frac{\partial F}{\partial y} = A(t) \frac{\partial \psi}{\partial v} H(t) \]

and

\[ \frac{\partial F}{\partial t} = A(t) \psi + xA(t) \frac{\partial \psi}{\partial u} H'(t) + yA(t) \frac{\partial \psi}{\partial v} H'(t) \]

Now eliminating \( \frac{\partial \psi}{\partial u} \) and \( \frac{\partial \psi}{\partial v} \) with the help of (2.24, b), (2.25, a), (2.25, b) and by substituting these values in (2.26), it becomes

\[ \frac{I}{A(t)} = \frac{I}{A(t)} F + x \frac{\partial F}{\partial x} H(t) + y \frac{\partial F}{\partial y} H'(t) \]

Equation (2.14) and (2.24, a) gives

\[ \frac{\partial F}{\partial t} = \sum_{n=0}^{\infty} \alpha_n t^{n+1} F + \sum_{n=0}^{\infty} \beta_n t^{n+1} \]

Since \( F = \sum_{n=0}^{\infty} q_n(x,y) t^n \), then the above equation becomes

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \alpha_n q_{n-k}(x,y) t^n \]

Hence,

\[ xq_n(x,y) + yq_n(x,y) - nq_n(x,y) = \sum_{n=0}^{\infty} q_n(x,y) t^n \]

This is the proof of theorem 7.

**Theorem 8**

If \( q_n(x,y) \) is defined by

\[ \sum_{n=0}^{\infty} q_n(x,y) = A(t) \psi \left[ xH(t) + g(t), yH(t) + g(t) \right] \]

and holding the conditions (2.7, a), (2.18, a), (2.18, b) and \( g(t) = \sum_{n=0}^{\infty} g_n t^{n+2} \),

then \( q_n(x,y) \) is a polynomial in \( x \) and \( y \) and is of the degree precisely \( 2n \) if and only if

\[ \sigma_r \neq 0 \text{ and } \nu_k \neq 0. \]

**Proof of Theorem 8**

Let

\[ q_n(x,y) = \sum_{r,k=0}^{n} S(n,k,r) x^r y^k \]

then (2.27) becomes

\[ A(t) \psi \left[ xH(t) + g(t), yH(t) + g(t) \right] = \sum_{n=0}^{\infty} S(n,r,k) x^r y^k t^n \]

By using the same process of theorem-6, we get

\[ A(t) \psi \left[ g(t), g(t) \right] \left[ H(t) \right] = \sum_{n=0}^{\infty} S(n,m_1,m_2) m_1! m_2! t^n \]

Now by applying (2.7, a), (2.18, a), (2.18, b) and (2.28) in (2.29) it yields
\[ A(t) \psi_{n^m}^{m_n} (g(t), g(t)) (H(t))^{m_0 + m_2} = a_0 \sigma_m \nu_m h_0^{m_0 + m_2} m_1! m_2! t^{m_0 + m_2} + \sum_{n=m_0 + m_1 + 1}^{\infty} C(n, m_1, m_2) t^n \]  

(2.30)

Here \( u = xH(t) + g(t) \) and \( v = yH(t) + g(t) \)

By comparing equation (2.29) and (2.30) we get

\[ a_0 \sigma_m \nu_m h_0^{m_0 + m_2} m_1! m_2! t^{m_0 + m_2} + \sum_{n=m_0 + m_1 + 1}^{\infty} C(n, m_1, m_2) t^n = \sum_{n=0}^{\infty} S(n, m_1, m_2) m_1! m_2! t^n \]  

(2.31)

For \( m_1 = m_2 \), (2.31) becomes

\[ a_0 \sigma_m \nu_m h_0^{2m} (m!)^2 t^{2m} + \sum_{n=0}^{\infty} C(n, m)(m!)^2 t^n = \sum_{n=0}^{\infty} S(n, m)(m!)^2 t^n \]  

(2.32)

On comparing the coefficient of \( t \), we get

\[ S(m, n) = 0 \text{ for } n < 2m \quad \text{and} \quad S(n, m) = a_0 \sigma_m \nu_m h_0^{2m} \text{ for } n = 2m. \]  

(2.33)

Hence, we arrive at conclusion that \( q_n (x, y) \) is a polynomial of degree precisely 2n if and only if \( \sigma_m \neq 0, \nu_m \neq 0 \) and \( a_0 h_0 \neq 0 \).

**THEOREM. 9**

If the polynomials \( q_n (x, y) \) defined by (2.27) and holding (2.7, a), (2.18, a), (2.18, b) and (2.28), where \( \sigma_r \neq 0, \nu_k \neq 0 \), then there exist sequence of numbers \( \alpha_k, \beta_k \) and \( \gamma_k \) such that,

\[ xq_{n_1} + yq_{n_2} - nq_n + \sum_{k=0}^{n-1} \alpha_k q_{(n-1-k)} + \sum_{k=0}^{n-1} (x\beta_k + \gamma_k) q_{(n-1-k)} + \sum_{k=0}^{n-1} (y\beta_k + \gamma_k) q_{(n-1-k)} = 0 \]

For \( n \geq 1 \)

**Proof of Theorem 9.**

Let \( \frac{tg'(t)}{H(t)} = \sum_{n=0}^{\infty} \frac{\psi_n t^{n+1}}{n+1} \)  

(2.34)

Put \( F = A(t) \psi \left[ xH(t) + g(t), yH(t) + g(t) \right] \)

Then \( \frac{\partial F}{\partial x} = A(t) H(t) \frac{\partial \psi}{\partial u} \) and \( \frac{\partial F}{\partial y} = A(t) H(t) \frac{\partial \psi}{\partial v} \)

Here \( u = xH(t) + g(t), v = yH(t) + g(t) \)

and \( \frac{\partial F}{\partial t} = A(t) \psi + A(t) \frac{\partial u}{\partial t} (xH'(t) + g'(t)) + A(t) \frac{\partial \psi}{\partial v} (yH'(t) + g'(t)) \)  

(2.35)

By eliminating the value of \( \psi, \frac{\partial \psi}{\partial u} \) and \( \frac{\partial \psi}{\partial v} \) and substituting these values on equation (2.35) we get

\[ \frac{\partial F}{\partial t} = \frac{tA'(t)}{A(t)} \frac{\partial F}{\partial H(t)} + t \frac{\partial F}{\partial H(t)} + \frac{\partial F}{\partial H(t)} + \frac{\partial F}{\partial H(t)} \]

now using equation (2.24, a), (2.24, b) and (2.34)

\[ \sum_{n=1}^{\infty} nq_n (x, y) t^n = \left[ \sum_{n=0}^{\infty} \alpha_n t^{n+1} \right] \left[ \sum_{n=0}^{\infty} q_n (x, y) t^n \right] + x \left[ 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \right] \left[ \sum_{n=0}^{\infty} q_n (x, y) t^n \right] \]
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\[ + \left( \sum_{n=0}^{\infty} \gamma_n t^{n+1} \right) \left[ \sum_{n=0}^{\infty} q_n (x, y) t^n \right] + y \left[ 1 + \sum_{n=0}^{\infty} \beta_n t^{n+1} \right] \left[ \sum_{n=0}^{\infty} q_n (x, y) t^n \right] + \sum_{n=0}^{\infty} \gamma_n t^{n+1} \left[ \sum_{n=0}^{\infty} q_n (x, y) t^n \right] \]

after simplifying, we get

\[ xq_{n+1} + yq_{n+1} - nq_{n+1} + \sum_{k=0}^{n-1} \alpha_k q_{(n-1-k)} + \sum_{k=0}^{n-1} (x\beta_k + y\gamma_k) q_{(n-1-k)} + \sum_{k=0}^{n-1} (y\beta_k + y\gamma_k) q_{(n-1-k)} = 0 \]

REFERENCES


Received: December, 2010