Finite Groups in which the Number of Subgroups of Possible Order is Less than or Equal 3

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Abstract

In this work, the finite groups in which the number of subgroups of possible order is less than or equal to 3 are determined. In addition, the complete classification of the finite $p$–groups in which the number of subgroups of possible order is 1 or $p + 1$. Based on the computational results, we make the following conjecture:

If the finite group $G$ has a subgroup of order $k$, then the number of subgroups of order $k$ in $G$ is not equal to 2.

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1 Introduction

The enumeration problem of finite $p$–groups is important in the research of finite $p$–groups. It includes two parts: (a) study the number of subgroups, elements and subsets of finite $p$–groups, (b) study the structure or properties of finite $p$–groups by means of the number of subgroups. Some remarkable counting theorems are made in this field.
Theorem 1.1 (L. Sylow (1872), or see [1]) Assume $G$ is a group of order $p^n$, $0 \leq k \leq n$. $s_k(G)$ denotes the number of subgroups of order $p^k$ of $G$. Then $s_k(G) \equiv 1 \pmod{p}$.

Theorem 1.2 (see [2] Kulakoff) Assume $G$ is a non-cyclic group of order $p^n$, $p > 2$. If $1 \leq k \leq n - 1$, then $s_k(G) \equiv 1 + p \pmod{p^2}$.

Theorem 1.3 (see [1]) Assume $G$ is a group of order $p^n$, $0 \leq k \leq n$. If $s_1(G) = 1$, then $G$ is a cyclic group, or a general quaternions group.

Theorem 1.4 (see [1]) Assume $G$ is a group of order $p^n$, $0 \leq k \leq n$. If $s_k(G) = 1$, $2 \leq k \leq n - 1$, then $G$ is a cyclic group.

Obviously, studying the structure of finite $p-$groups in which the numbers of subgroups of possible order is a pre-given figure is an interesting question. In fact, by P. Hall’s enumeration principle, groups of order $p^n$ in which the number of nontrivial subgroups of possible order is equal to $1 + p$ are classified [3]. In this work, we class finite groups (not only finite $p-$groups) in which the number of subgroups of possible order is less than or equal 3. In addition, we also give out the complete classification of finite $p-$groups in which the number of subgroups of possible order is 1 or $p + 1$.

For convenience, let $s_k(G)$ denote the number of subgroups of order $p^k$ of a finite $p-$group $G$; $n_p$ to denote the number of the Sylow $p-$subgroup of a finite non $p-$group; $n(G)$ to denote the set of the number of subgroups of possible order of a finite group.

The notations and symbols in this paper are referred to [4].

2 Preliminary Notes

To draw the conclusion, some lemmas are firstly given as follows.

Lemma 2.1 (see [4]) Assume $G$ is not a finite abelian group and all its Sylow subgroups are cyclic. Then $G$ is

$$G = < a, b >, a^m = b^n = 1, b^{-1}ab = a^r, ((r - 1)n, m) = 1, r^n \equiv 1 \pmod{m}, |G| = nm.$$ where $m, n, r$ are positive integers.

Lemma 2.2 (see [4]) Assume $G$ is a finite which is not a $p-$group and all its subgroups are cyclic. Then $G$ is

$$G = < a, b >, a^p = b^q = 1, b^{-1}ab = a^r, r \not\equiv 1 \pmod{p}, r^q \equiv 1 \pmod{p}$$ where $p, q$ are pairwise different prime numbers and $m, r$ are positive integers.

Lemma 2.3 Assume $G$ is a finite $p-$group. If the number of subgroups of possible order of $G$ is less or equal to 3, then $G$ is a cyclic group, or a non cyclic 2-group.
The number of subgroups of finite group

Proof Suppose that $G$ is not a cyclic $p$–group and $p \geq 3$, then $G$ must have an abelian subgroup with type $(p, p)$. Thus, the number of subgroups of order $p$ in $G$ is more than $p+1 > 3$, which is contradicted with the assumption. Hence $G$ is a cyclic group, or a non cyclic $2$–group.

Lemma 2.4 Assume $G$ is a finite which is not a $p$–group. If the number of subgroups possible order of $G$ is less or equal to 3, then all Sylow subgroups of $G$ with odd order are cyclic and normal.

Proof Suppose $n_p > 1$ and $p \geq 3$. If $n_p = 2$, then $n_p \equiv 1(mod \ p)$, which is contradicted with Sylow Theorem. If $n_p = 3$, then $n_p \equiv 1(mod \ p)$, hence $p = 2$, which is contradicted with assumption $p \geq 3$. Thus, all Sylow subgroups of $G$ with odd order are normal. By lemma2.3, all Sylow subgroups of $G$ with odd order are cyclic.

3 Main Results

Theorem 3.1 Assume $G$ is a finite $p$–group. If $n(G) = \{1, p + 1\}$, then $G$ is one of the following cases:

(i) $Q_8$;
(ii) an abelian group with type $(2^{n-1}, 2)$ where $n \geq 2$;
(iii) $G = \langle a, b \rangle, a^{p^n-1} = 1, b^p = 1, b^{-1}ab = a^{1+p^n-2}$, where $p$ is odd prime and $n \geq 3$;
(iv) $G = \langle a, b \rangle, a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+2^{n-2}}$, where $p = 2$, and $n \geq 4$.

Proof Now we prove the theorem by three cases.

(i) If $s_1(G) = 1$, by theorem1.3, $G$ is a cyclic group, or a general quaternions group. But $G$ has $p + 1$ subgroups of order $p^k$, so $G$ is a general quaternions group, not a cyclic group. By the theorem 1 of [3], we know $G$ is $Q_8$.

(ii) If $k$ is a positive integer and $n > k \geq 2$, $s_k(G) = 1$, by theorem 1.4, $G$ is a cyclic group. Thus $n(G) = \{1\}$, which is contradicted with assumption.

(iii) If the number of nontrivial subgroups of possible order of $G$ is equal to $1 + p$, by the theorem 2 of [3], $G$ is one case of (2), (3), or (4).

Theorem 3.2 If the number of subgroups of possible order of a finite group $G$ is less than or equal to 3, then $G$ is one of the following cases:

I. $n(G) = \{1\}$
   (1) $G$ is a finite cyclic group.
II. $n(G) = \{1, 3\}$ and $G$ is a finite $p$–group
   (2) $Q_8$;
   (3) an abelian group with type $(2^{n-1}, 2)$;
   (4) $G = \langle a, b \rangle, a^{2^{n-1}} = 1, b^2 = 1, b^{-1}ab = a^{1+2^{n-2}}$, where $n \geq 4$.
III. $n(G) = \{1, 3\}$, $G$ is a finite but not a $p$–group and the Sylow 2–subgroup of $G$ is normal.
(5) \(Q_8 \times < a >, 2 \nmid o(a)\);
(6) \(P \times < u >, 2 \nmid o(u)\), \(P\) is an abelian group with type \((2^{n-1}, 2)\);
(7) \(P \times < v >, 2 \nmid o(v)\), \(P\) is one group of (4).

IV. \(n(G) = \{1, 3\}\), \(G\) is a finite but not a \(p\)-group and the Sylow 2-subgroup of \(G\) is not normal

(8) \(G = (P \times P_1) \times < u >, \quad P = < a >, \quad P_1 = < b >, \quad o(a) = 2^n, \quad n \geq 1, \quad o(b) = 3, \quad 2 \nmid o(u), \quad 3 \nmid o(u)\).

Where \(n(G)\) is the set of the number of subgroups of \(G\).

**Proof** Now we prove the theorem case by case of possible order of \(G\).

Case1 \(n(G) = \{1\}\). This case is trivial and it is easy to know \(G\) is a finite cyclic group.

Case2 \(n(G) = \{1, 2\}\). We assert that no finite group \(G\) satisfies \(n(G) = \{1, 2\}\).

If \(G\) is a \(p\)-group, by lemma2.3, \(G\) is a cyclic group, or a non cyclic 2-group.
By case1, \(G\) is a non cyclic 2-group. Hence it exists \(k, 1 \leq k < n, s_k(G) = 2\).
By theorem1.1, which is impossible.

If Assume \(G\) is a finite but not a \(p\)-group, by lemma2.4, all Sylow subgroups of \(G\) with odd order are cyclic and normal. From the proof above, we know the Sylow 2-subgroup of \(G\) is cyclic and normal. Thus \(G\) is a cyclic group. By case1, which is contradicted with \(n(G) = \{1, 2\}\).

Case3 \(n(G) = \{1, 3\}\).

If \(G\) is a finite \(p\)-group, by theorem3.1, \(G\) is one of following three types:
(2) \(Q_8\);
(3) an abelian group with type \((2^{n-1}, 2)\);
(4) \(G = < a, b >, \quad a^{2^{n-1}} = 1, \quad b^2 = 1, \quad b^{-1}ab = a^{1+2^{n-2}}, \quad \text{where } n \geq 4\).

If \(G\) is a finite but not a \(p\)-group, by lemma2.4, all Sylow subgroups of \(G\) with odd order are cyclic and normal. Next part, we prove the case on whether the Sylow 2-subgroup of \(G\) is normal or not.

i) If the Sylow 2-subgroup of \(G\) is normal, by case1, the Sylow 2-subgroup of \(G\) is not a cyclic subgroup. From the proof above, the Sylow 2-subgroup of \(G\) is one case of (2), (3), or (4). Thus, \(G\) is one of the following cases:
(5) \(Q_8 \times < a >, \quad 2 \nmid o(a)\);
(6) \(P \times < u >, \quad 2 \nmid o(u)\), \(P\) is an abelian group with type \((2^{n-1}, 2)\);
(7) \(P \times < v >, \quad 2 \nmid o(v)\), \(P\) is one group of (4).

ii) If the Sylow 2-subgroup of \(G\) is non-normal, then we assert that the Sylow 2-subgroup of \(G\) is a cyclic subgroup. Suppose that the Sylow 2-subgroup of \(G\) is not a cyclic subgroup. From the proof above, we know that the Sylow 2-subgroup of \(G\) is isomorphic to one case of (2), (3), or (4). So we can assume that the three types of the Sylow 2-subgroups of \(G\) are \(P_1, P_2,\) and \(P_3\). Clearly, they are isomorphic to each other. If they are isomorphic to \(Q_8\), then each of \(P_1, P_2,\) and \(P_3\) has three subgroups of order 4, respectively. By the \(n(G) = \{1, 3\}\), we know the subgroups of order 4 are identi-
cal. Thus, \( P_1, P_2, \) and \( P_3 \) are the same group, which is contradicted with the assumption.

Similarly, if \( P_1, P_2, \) and \( P_3 \) are isomorphic to (3) or (4), we can also get a contradiction.

Let \( G = P \times T \), \( P \) is a Sylow 2–subgroup of \( G \), \( T = P_1 \times P_2 \times \cdots \times P_t \), \( t \geq 1 \), \( P_i \) is a Sylow \( p_i \)–subgroup of \( G \), \( p_i \) is an odd prime, \( i = 1, 2, \cdots, t \). Without loss of generality, \( P_1 \) is a Sylow 3–subgroup of \( G \). Assume \( P \) acts nontrivially on \( P_1, P_2 \), then \( < P, P_1 > = P \times P_1, < P, P_2 > = P \times P_2 \) have three Sylow 2–subgroups, respectively. Because \( n_2(P \times P_1) = 3 \), \( n_2(P \times P_2) = 3 \), so \( 3 \mid |P_1|, 3 \mid |P_2| \). This means that \( P_1, P_2 \) are Sylow 3–subgroup of \( G \), which is a contradiction. So \( P \) just acts nontrivially on \( P_1 \) and trivially on \( P_2, \cdots, P_t \), which means \( G = \langle P \times P_1 \rangle \times P_2 \times \cdots \times P_t \).

Next considering the structure of \( P \times P_1 \), and prove \( n(P \times P_1) = \{1, 3\} \). Let \( G = P \times P_1, P = \langle a >, P_1 = \langle b >, o(a) = 2^n, o(b) = 3^m, n, m \geq 1 \). Because \( P_1 \) is a normal subgroup of \( G \) and \( P \) is not, hence \( a^{-1}ba = b^r \), and \( r \not\equiv 1(\text{mod } 3^m) \).

Further more \( n_2(G) = 3 \), \( |G : N_G(P)| = 3 \), so \( |N_G(P)| = 2^n 3^{m-1} \), which means \( N_G(P) = P \times < b^3 > \).

Now we prove \( m = 1 \). Assume \( m > 1 \), then \( b^3 = a^{-1}b^3a = (a^{-1}ba)^3 \), \( b^{3(r-1)} = 1 \). Thus \( r \equiv 1(\text{mod } 3^{m-1}) \). By lemma2.1, \( ((r-1)2^n, (3^m)) = 1 \). It means \( m = 1 \), which is a contradiction.

Now we prove \( r = -1 \). Because \( r \not\equiv 1(\text{mod } 3^m) \), hence \( r \equiv 0 \), or \( -1(\text{mod } 3^m) \). If \( r \equiv 0(\text{mod } 3^m) \), then \( a^{-1}ba = b^r = 1 \), which is a contradiction. Thus, \( r \equiv -1(\text{mod } 3^m) \), and we can obtain \( r = -1 \).

Let \( G = \langle a, b >, o(a) = 2^n, o(b) = 3, a^{-1}ba = b^{-1}, n \geq 1 \). By lemma2.2, all subgroups of \( G \) are cyclic, and also \( a^{-2}ba^2 = a^{-1}(a^{-1}ba)a = a^{-1}b^2a = b, (ab)^2 = a(ba)b = aab^{-1}b = a^2, (ab^2)^2 = ab^2ab^2 = ab(ba)bb = abab^{-1}bb = (ab)^2 = a^2 \), these mean \( o(a^2b) = 2^n 3, o(a) = o(ab^2) = o(ab) = 2^n \). Thus,

the maximal subgroups of \( G \) are \( \langle a^2b >, < a >, < ab^2 >, < ab > \), and also \( (ab^2)^2 = (ab)^2 = a^2 \), hence \( \Phi(G) = \langle a^2 > \). From all discussion above, we can get the subgroups of \( G \) of order \( 2^{n-1}3 \) are \( \langle a^2b >, i = 1, 2, \cdots, n \); the subgroups of \( G \) of order \( 2^n \) are \( \langle a >, < ab^2 >, < ab > \); the subgroups of \( G \) of order \( 2^i \) are \( \langle a^i >, i = 1, 2, \cdots, n \). Thus, \( n(P_1 \times P) = \{1, 3\} \).

Case4 \( n(G) = \{1, 2, 3\} \). We assert that no finite group \( G \) satisfies \( n(G) = \{1, 2, 3\} \).

If \( G \) is a \( p \)–group, by lemma2.3, \( G \) is a cyclic group, or a non cyclic 2–group. By case1, we know \( G \) is a non cyclic 2–group. Thus, it exists \( k, 1 \leq k < n \), and \( s_k(G) = 2 \). By theorem1.1, this is impossible.

If \( G \) is a finite which is not a \( p \)–group, by lemma2.4, all Sylow subgroups of \( G \) of odd order are cyclic and normal. Next we prove the case on whether the Sylow 2–subgroup of \( G \) is normal or not.

i) If the Sylow 2–subgroup of \( G \) is normal, noting \( P \), by discussion of case3 above, we know that \( n(P) = \{1, 3\} \). Hence \( n(G) = \{1, 3\} \), which is contra-
dicted with \( n(G) = \{1, 2, 3\} \).

ii) If the Sylow 2−subgroup of \( G \) is not normal, then \( n_2 = 2 \), or 3. If \( n_2 = 2 \), then \( n_2 = 2 \equiv 0 \pmod{2} \), which is a contraction. Hence \( n_2 = 3 \). By discussion of case 3 above, we know that \( n(G) = \{1, 3\} \), which is contradicted with \( n(G) = \{1, 2, 3\} \).

We can easily justify that all groups we obtain satisfy \( n(G) = \{1, 3\} \).

From discussion of case 2 and case 3, we know that no finite group satisfies \( n(G) = \{1, 2\} \), and \( n(G) = \{1, 2, 3\} \), but existing finite groups which satisfy \( n(G) = \{1, 3\} \). Thus we make a open problem in the end of this work as follows:

*If the finite group \( G \) has a subgroups with order \( k \), then the number of subgroup with order \( k \) is not equal to 2.*

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**References**


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