A Note on Piecewise-Koszul Complexes\textsuperscript{1}

Pan Yuan

Yiwu Industrial and Commercial College
Yiwu, Zhejiang 322000 P.R. China

Abstract. The notion of piecewise-Koszul complex was first introduced by Lü and Si in [2]. In order to make the complex exact, the authors defined the so-called special piecewise-Koszul algebras. In this short note, we prove that for a positively graded algebra $A$, $A$ is a special piecewise-Koszul algebra if and only if $A$ is a Koszul algebra, which shows that we have to change the definition of piecewise-Koszul complex for a non-Koszul piecewise-Koszul algebra.

Mathematics Subject Classification: 16S37, 16W50

Keywords: Koszul algebras, piecewise-Koszul algebras, piecewise-Koszul complexes

1. Introduction

Piecewise-Koszul algebras, first introduced by Lü, He and Lu in [1], were another quadratic homogeneous positively graded algebras in general. Recently, Lü and Si defined the so-called piecewise-Koszul complex related to a piecewise-Koszul algebra. Further, in order to make the complex exact, the authors defined the so-called special piecewise-Koszul algebras. In this short note, we prove that $A$ is a special piecewise-Koszul algebra if and only if $A$ is a Koszul algebra, which shows that we have to change the definition of piecewise-Koszul complex in [2] for a non-Koszul piecewise-Koszul algebra.

Now let us recall some notations and definitions.

Throughout, let $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and natural numbers, $\mathbb{K}$ denotes a fixed field, and all the graded $\mathbb{K}$-algebras $A = \bigoplus_{i \geq 0} A_i$ are assumed with the following properties:

(a) $A_0 = \mathbb{K} \times \mathbb{K} \times \cdots \times \mathbb{K}$, finite copies of $\mathbb{K}$;

(b) $A$ is generated in degrees 0 and 1; that is, $A_i \cdot A_j = A_{i+j}$ for all $0 \leq i, j < \infty$;

\textsuperscript{1}Research is supported by Yiwu Industrial And Commercial College(Grant 2011019) and the foundation of Educational Department of Zhejiang Province in China(Grant Y201016432)
Let \( M = \bigoplus_{i \geq 0} M_i \) be a finitely generated graded \( A \)-module. We call \( M \) a \textit{piecewise-Koszul module} provided that \( M \) admits a minimal graded projective resolution
\[
\mathbf{F} : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0
\]
with each \( F_n \) generated in degree \( \delta^d(n) \), where the set function \( \delta^d : \mathbb{N} \to \mathbb{N} \) is defined by
\[
\delta^d_p(n) = \begin{cases} 
\frac{nd}{p}, & \text{if } n \equiv 0(p), \\
\frac{(n-1)d}{p} + 1, & \text{if } n \equiv 1(p), \\
\cdots & \text{if } n \equiv p-1(p).
\end{cases}
\]
and \( d \geq p > 2 \).

In particular, if the trivial \( A \)-module \( A_0 \) is a piecewise-Koszul module, then \( A = \bigoplus_{i \geq 0} A_i \) is called a \textit{piecewise-Koszul algebra}.

**Definition 1.2.** [2] Let \( A = T_{A_0}(A_1)/\langle R \rangle \) be a quadratic algebra, define the so called \textit{piecewise-Koszul complex}
\[
\mathcal{P}K^* : \cdots \to \mathcal{P}K^{pm} \to \mathcal{P}K^{pm+1} \to \cdots \to \mathcal{P}K^p \to \mathcal{P}K^{p+1} \to \cdots \to \mathcal{P}K^1 \to \mathcal{P}K^0
\]
as follows.

For any \( i \geq 0 \), \( \mathcal{P}K^i \) is a graded projective \( A \)-module given by
\[
\mathcal{P}K^i = A \otimes_{A_0} \mathcal{P}K^i_{\delta^p(i)},
\]
where \( \mathcal{P}K^i_{\delta^p(i)} \) is an \( A_0 \)-\( A_0 \)-bimodule concentrated in degree \( \delta^p(i) \), i.e., for \( i \geq 3 \),
\[
\mathcal{P}K^i_{\delta^p(i)} = \bigcap_{u+v+2 = \delta^p(i)} A_1^u \otimes R \otimes A_1^v \subseteq A_1^\otimes_{\delta^p(i)}.
\]
In particular, we have
\[
\mathcal{P}K^0 = A, \quad \mathcal{P}K^1 = A \otimes A_1, \quad \mathcal{P}K^2 = A \otimes R.
\]
The differential of \( \mathcal{P}K^* \), \( \partial^i : \mathcal{P}K^i \to \mathcal{P}K^{i-1} \) is the restriction of the map
\[
\partial^i : A \otimes A_1^\otimes_{\delta^p(i)} \to A \otimes A_1^\otimes_{\delta^p(i-1)}
\]
via
\[
a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(i)} \mapsto aa_1 \cdots a_{\delta^p(i)-\delta^p(i-1)} \otimes a_{\delta^p(i)-\delta^p(i-1)+1} \otimes \cdots \otimes a_{\delta^p(i)}.
\]
More precisely,
\[
\partial^{pm}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm)}) = aa_1 a_2 \cdots a_{d-p+1} \otimes a_{d-p+2} \otimes \cdots \otimes a_{\delta^p(pm)};
\]
\[
\partial^{pm+1}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm+1)}) = aa_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm+1)};
\]
\[
\partial^{pm+2}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm+2)}) = aa_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm+2)};
\]
\[
\cdots
\]
\[
\partial^{pm+p-1}(a \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm+p-1)}) = aa_1 \otimes a_2 \otimes \cdots \otimes a_{\delta^p(pm+p-1)},
\]
where \( a \in A \) and \( a_i \in A_1, i = 1, 2, \ldots \).
Definition 1.3. [2] Let $A$ be a piecewise-Koszul algebra, writing $A$ as $T_{A_0}(A_1)/\langle R \rangle$. If we have

$$(A_1^{\otimes n} \otimes R) \bigcap (R \otimes A_1^{\otimes n}) = \bigcap_{i+j=n} (A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j})$$

for all $1 \leq n \leq d-2$, then $A$ will be called a special piecewise-Koszul algebra.

2. The main result and proof

We begin with recalling one of the main results of [2], we also give the proof here for the completeness.

Lemma 2.1. Let $A$ be a special piecewise-Koszul algebra and $E(A)$ its Yoneda-Ext algebra. Let $B = \bigoplus_{i \geq 0, j \in \mathbb{Z}} B^i_j$, where $B^i_j = B^{i}_{\delta^d_p(i)}$ and $B^i_j = 0$ if $j \neq \delta^d_p(i)$. Then $E(A) \cong B$ as bigraded $\mathbb{K}$-algebras.

Proof. Firstly we claim:

$$\text{Ext}^i_A(A_0, A_0) = B^i_{\delta^d_p(i)}, \quad \forall \ i \geq 0.$$ 

In fact, since $A$ is a special piecewise-Koszul algebra, by Theorem 2.5 of [2], the piecewise-Koszul complex $PK^*$ of $A$ is a projective resolution of the trivial $A$-module $A_0$. Hence

$$\text{Ext}^i_A(A_0, A_0) = \frac{\ker(\text{Hom}_A(PK^i, A_0) \to \text{Hom}_A(PK^{i+1}, A_0))}{\text{Im}(\text{Hom}_A(PK^{i+1}, A_0) \to \text{Hom}_A(PK^i, A_0))}$$

$$= \text{Hom}_A(PK^i, A_0) = \text{Hom}_A(A \otimes_{A_0} PK^i_{\delta^d_p(i)}, A_0)$$

$$= \text{Hom}_{A_0}(PK^i_{\delta^d_p(i)}, \text{Hom}_A(A, A_0))$$

$$= \text{Hom}_{A_0}(PK^i_{\delta^d_p(i)}, A_0)$$

$$= \text{Hom}_{A_0}((A^i_{\delta^d_p(i)})^*, A_0)$$

$$= A^i_{\delta^d_p(i)} = B^i_{\delta^d_p(i)}.$$ 

In order to finish the proof, define a multiplication $\ast$ on $B$ as follows: for $\psi \in B^i$ and $\varphi \in B^j$,

$$\psi \ast \varphi = \begin{cases} 
\psi \cdot \varphi, & \text{at least one of } i \text{ and } j \text{ is of the form } kp, \, k \in \mathbb{Z}, \\
0, & \text{otherwise}, 
\end{cases}$$

where $\cdot$ is the product of the dual algebra $A^i$. Under the multiplication $\ast$, it is easy to see that $B$ is a bigraded algebra. Note that we have the following isomorphisms,

$$A^i_{\delta^d_p(n)} = (PK^n_{\delta^d_p(n)})^*, \quad \text{Ext}^i_A(A_0, A_0) = A^i_{\delta^d_p(n)}$$

and

$$\text{Ext}^n_A(A_0, A_0) = \text{Hom}_A(A \otimes_{A_0} PK^n_{\delta^d_p(n)}, A_0).$$
Let $\varphi \in \text{Ext}_A^n(A_0, A_0) = A^1_{d_p(n)}$, $\psi \in \text{Ext}_A^m(A_0, A_0) = A^1_{d_p(m)}$. Let $\varphi \cdot : A \otimes_{A_0} \mathcal{P}K^{n}_{d_p(n)} \rightarrow A_0$ and $\psi \cdot : A \otimes_{A_0} \mathcal{P}K^{m}_{d_p(m)} \rightarrow A_0$ be the $A$-module morphisms induced by $\varphi$ and $\psi$, respectively. Now showing that

$$\psi \star \varphi = \psi \cdot \varphi,$$

where $\star$ denotes the product of $\text{Ext}_A^n(A_0, A_0)$, $\cdot$ the product of $A^1$. Now consider the following diagram

$$
\begin{array}{cccccc}
\cdots \rightarrow & A \otimes \mathcal{P}K^{n+m}_{d_p(n+m)} & \cdots & \rightarrow & A \otimes \mathcal{P}K^m_{d_p(m)} & \cdots \\
\downarrow \varphi_m & \downarrow \varphi & \cdots & \varphi & \downarrow \varphi & \cdots \\
\cdots \rightarrow & A \otimes \mathcal{P}K^m_{d_p(m)} & \cdots & \rightarrow & A \otimes \mathcal{P}K^0_{d_p(0)} & \rightarrow A_0 \rightarrow 0
\end{array}
$$

where $\varphi_i$ for $i = 0, 1, \ldots, m$ is defined as follows.

(i) If $n = pk$. Define

$$\varphi_i(a \otimes v_1 \otimes \cdots \otimes v_{d_p(n+m)}) = a \otimes v_1 \otimes \cdots \otimes v_{d_p(i)} \varphi(v_{d_p(i)+1} \otimes \cdots \otimes v_{d_p(n+l)}),$$

where $l = 0, 1, \ldots, m$. Clearly, $\partial'\varphi_i = \varphi_{i-1}\partial^{n+l}$. Thus,

$$\begin{align*}
\varphi_i(a \otimes v_1 \otimes \cdots \otimes v_{d_p(n+m)}) &= \varphi(a \otimes v_1 \otimes \cdots \otimes v_{d_p(m)}) \varphi(v_{d_p(m)+1} \otimes \cdots \otimes v_{d_p(n)}) \\
&= a \otimes \varphi(v_1 \otimes \cdots \otimes v_{d_p(m)}) \varphi(v_{d_p(m)+1} \otimes \cdots \otimes v_{d_p(n)}) \\
&= a \varphi \cdot \varphi(v_1 \otimes \cdots \otimes v_{d_p(n+m)}).
\end{align*}$$

(ii) If $n = pk + i$, $i = 1, 2, \ldots, p - 1$. If $l = pk' + j$, $j = 1, 2, \ldots, p - 1$. Then define

$$\varphi_i(a \otimes v_1 \otimes \cdots \otimes v_{d_p(n+l)}) = av_1 \cdots v_{d-p+1} \otimes \cdots \otimes v_{d_p(n+l)-d_p(n)} \varphi(v_{d_p(n+l)-d_p(n)+1} \otimes \cdots \otimes v_{d_p(n+l)}).$$

Similarly, $\partial'\varphi_i = \varphi_{i-1}\partial^{n+l}$. If $\psi \star \varphi \neq 0$, then $\delta_p^i(m + n) = \delta_p^i(m) + \delta_p^i(n)$. Since $n = pk + i$, $i = 1, 2, \ldots, p - 1$, clearly $m = pk'$. By the computations similar to (i), we have $\psi \star \varphi = \psi \cdot \varphi$. Therefore, $E(A) \cong B$ as bigraded $K$-algebras.

\begin{lemma} \text{([3])} \label{lemma:2.2}
Let $A$ be a positively graded algebra and $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated module over $A$. Then we have the following isomorphisms:

$$
\bigoplus_{i \geq 0} \text{Ext}^i_A(A_0, A_0) \cong (qA)_i, \quad \bigoplus_{i \geq 0} \text{Ext}^i_A(M, A_0) \cong (qM)_i^{qA},
$$

where $qA$ and $qM$ are the quadratic parts of $A$ and $M$, respectively.
\end{lemma}
Proposition 2.3. Let $A$ be a piecewise-Koszul algebra and $E(A)$ its Yoneda-Ext algebra. Then

$$A^! \cong \langle \text{Ext}_A^1(A_0, A_0) \rangle = \langle \text{Ext}_A^1(A_0, A_0)_1 \rangle.$$

Proof. By the definition of piecewise-Koszul algebra, we have that $A$ is a quadratic algebra. Thus $qA = A$, which implies easily that $(qA)^! = A^!$. By Lemma 2.2, we have $A^! \cong \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0)$. Note again that $A$ is piecewise-Koszul, we have $\text{Ext}_A^1(A_0, A_0) = \text{Ext}_A^1(A_0, A_0)_1$. Therefore, we finish the proof. \hfill \Box

Now we can state and prove our main result:

Theorem 2.4. Let $A = \bigoplus_{i \geq 0} A_i$ be a graded algebra. Then $A$ is a special piecewise-Koszul algebra if and only if $A$ is a Koszul algebra.

Proof. ($\Leftarrow$) By Lemma 2.4 of [7], for any Koszul algebra $A = T_{A_0}(A_1)/\langle R \rangle$, we have

$$(A_1^{\otimes n} \otimes R) \cap (R \otimes A_1^{\otimes n}) = \bigcap_{i+j=n} (A_1^{\otimes i} \otimes R \otimes A_1^{\otimes j})$$

for all $n \geq 0$. That is, Koszul algebras are certainly special piecewise-Koszul algebras.

($\Rightarrow$) Suppose that $A$ is a special piecewise-Koszul algebra, then by Lemma 2.1, we have

$$\bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0) \cong \bigoplus_{i \geq 0} A^!_{d^p(i)}$$

as bigraded algebras. Note that $A$ is a piecewise-Koszul algebra, by Lemma 2.2, we have

$$A^! \cong \langle \text{Ext}_A^1(A_0, A_0) \rangle = \langle \text{Ext}_A^1(A_0, A_0)_1 \rangle.$$

Note also that

$$\bigoplus_{i \geq 0} A^!_{d^p(i)} \subset A^!$$

and

$$\bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0) \subset \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0),$$

which implies that $A^! \cong \bigoplus_{i \geq 0} A^!_{d^p(i)}$ as graded algebras, which forces $d^p(i) = i$ for all $i \geq 0$. That is, $A$ is a Koszul algebra. \hfill \Box

Remark 2.5. Theorem 2.4 tells us that for a non-Koszul piecewise-Koszul algebra $A$, if we adopt the Definition 1.2, we can’t get the exactness of piecewise-Koszul complex $\mathcal{P}K^* \to A_0 \to 0$. That is, we have to change Definition 1.2 for the case of a non-Koszul piecewise-Koszul algebra.


References


Received: December, 2010