On Special Types of Putcha Semigroups whose Subgroups Belong to a Given Variety of Groups\(^1\)

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Abstract

A semigroup \(S\) is called a Putcha semigroup if, for every \(a, b \in S\), the assumption \(b \in S^1aS^1\) implies \(b^m \in S^1a^2S^1\) for some positive integer \(m\).

In this paper we characterize some special types of Putcha semigroups whose subgroups belong to a given variety of groups.

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1 Introduction

For an arbitrary semigroup \(S\), let \(E(S)\) denote the set of all idempotent elements of \(S\). If \(e \in E(S)\) then \(eSe\) is the greatest submonoid and \(G_e = \{x \in eSe : (\exists y \in eSe) \ xy = yx = e\}\) is the greatest subgroup of \(S\) in which \(e\) is the identity element. It is known that, for every \(e, f \in E(S)\), either \(G_e \cap G_f = \emptyset\) (if \(e \neq f\)) or \(G_e = G_f\) (if \(e = f\)).

If \(H\) is a pseudovariety of finite groups, then the set \(\overline{H}\) of all finite semigroups whose subgroups belong to \(H\) is also a pseudovariety (see [5]). In [1], the authors were concerned with the question of how to describe \(\overline{H}\) syntactically, given a good syntactic description of the pseudovariety \(H\).

In our present paper we concentrate our attention to some special types of not necessarily finite semigroups whose subgroups belong to a given variety of groups. We characterize them by the help of their least semilattice congruence.

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A semigroup $S$ is called a Putcha semigroup if, for every $a, b \in S$, the assumption $b \in S^1aS^1$ implies $b^m \in S^1a^2S^1$ for some positive integer $m$; with other words, the assumption that $a$ divides $b$ implies that $a^2$ divides some power of $b$.

In this paper we deal with some special types of Putcha semigroups whose subgroups belong to a given variety of groups.

2 Preliminaries

A semigroup $S$ is called a band if every element of $S$ is an idempotent element. A semigroup $L$ is called a left zero semigroup if it satisfies the identity $ab = a$ ($a, b \in L$). The notion of the right zero semigroup is the dual of the notion of the left zero one. A semigroup is called a rectangular band if it is a direct product of a left zero semigroup and a right zero semigroup.

For an arbitrary semigroup $S$, the set $E(S)$ of all idempotent elements of $S$ is a partially ordered set under the following ordering: $e \leq f$ if and only if $ef = fe = e$ ($e, f \in E(S)$). A semigroup $S$ is called a completely simple semigroup if it is simple and contains an idempotent element which is minimal in the above ordering of the idempotents of $S$.

**Result 2.1** (Corollary 2.52b of [3]) Every completely simple semigroup $S$ is a rectangular band $B = I \times J$ of groups $G_{i,j}$ ($i \in I, j \in J$), that is, $S$ is a union of disjoint subgroups $G_{i,j}$ ($i \in I, j \in J$) such that $G_{i,j}G_{m,n} \subseteq G_{i,n}$ for every $i, m \in I$ and $j, n \in J$.

A semigroup $S$ is called a rectangular group if it is a direct product of a rectangular band and a group $G$; if $G$ is commutative then $S$ is called a rectangular abelian group.

A commutative band is called a semilattice. A congruence $\alpha$ of a semigroup $S$ is called a semilattice congruence if the factor semigroup $I = S/\alpha$ is a semilattice. In this case the $\alpha$-classes $S_i$ ($i \in I$) are subsemigroups of $S$; we also say that $S$ is a semilattice $I$ of the subsemigroups $S_i$ ($i \in I$). A semigroup $S$ is called semilattice indecomposable if the universal relation of $S$ is the only semilattice congruence of $S$.

**Result 2.2** (Proposition I.8.3, Corollary I.8.4 and Lemma L.8.6 of [11]) Every semigroup has a least semilattice congruence $\eta$; the $\eta$-classes are semilattice indecomposable semigroups. With other words, every semigroup is decomposable into a semilattice of semilattice indecomposable semigroups.

A semigroup $S$ is called an archimedean semigroup if, for every element $a, b \in S$, there are positive integers $m$ and $n$ such that $a^n \in SbS$ and $b^m \in SaS$;
with other words, for every two elements of $S$, both of them divide some power of the other. It is known that every archimedean semigroup is semilattice indecomposable (see, for example, the proof of Theorem 2.1 of [7]).

**Result 2.3** (Theorem 2.1 of [12] or Theorem 2.1 of [7]) A semigroup $S$ is a semilattice of archimedean semigroups if and only if it is a Putcha semigroup. In such a case the corresponding semilattice congruence equals $\eta = \{(a, b) \in S \times S : a^n \in SbS, b^m \in SaS$ for some positive integers $m, n\}$ and is the least semilattice congruence on $S$.

**Result 2.4** (Theorem 3.2 of [4] or Theorem 2.2 of [7]) A semigroup $S$ is archimedean and contains at least one idempotent element if and only if it is an ideal extension of a simple semigroup containing an idempotent element by a nil semigroup.

### 3 The least semilattice congruence

Let $H$ denote a given variety of groups. A congruence $\alpha$ of a semigroup $S$ will be called an $LH$-congruence if every $\alpha$-class $T$ that is a subsemigroup of $S$ contains an idempotent element, and the local submonoid $eTe$ belongs to the variety $H$ for every idempotent element $e$ of $T$.

**Theorem 3.1** Let $H$ be a given variety of groups and $S$ a semigroup which is a semilattice $Y$ of subsemigroups $S_\alpha$ ($\alpha \in Y$) such that, for every $\alpha \in Y$, the subsemigroup $S_\alpha$ is an ideal extension of a completely simple semigroup by a nil semigroup. Then the subgroups of $S$ belong to $H$ if and only if the least semilattice congruence of $S$ is a maximal $LH$-congruence of $S$.

**Proof.** Let $S$ be a semigroup which is a semilattice $Y$ of subsemigroups $S_\alpha$ ($\alpha \in Y$) such that, for every $\alpha \in Y$, the subsemigroup $S_\alpha$ is an ideal extension of a completely simple semigroup $K_\alpha$ by a nil semigroup. It is clear that the subsemigroups $S_\alpha$ ($\alpha \in Y$) are archimedean semigroups. Thus, by Result 2.3, $S$ is a Putcha semigroup and the subsemigroups $S_\alpha$ ($\alpha \in Y$) are the $\eta$-classes of $S$, where $\eta$ denotes the least semilattice congruence of $S$.

Let $H$ be a given variety of groups. Assume that the subgroups of $S$ belong to $H$. We show that $\eta$ is a maximal $LH$ congruence of $S$. Let $\alpha \in Y$ be arbitrary. As $K_\alpha$ is completely simple, $S_\alpha$ contains an idempotent element. We show that $eS_\alpha e \in H$. By Result 2.1, $K_\alpha$ is a rectangular band $B = I \times J$ of its maximal subgroups $G_{i,j}$ ($i \in I, j \in J$), that is, $K_\alpha$ is a union of disjoint subgroups $G_{i,j}$ ($i \in I, j \in J$) such that $G_{i,j}G_{m,n} \subseteq G_{i,j}$ for every $i, m \in I$ and $j, n \in J$. The subgroups $G_{i,j}$ ($i \in I, j \in J$) are maximal in $K_\alpha$ and also in $S_\alpha$, because the Rees factor semigroup of $S_\alpha$ defined by $K_\alpha$ is nil. By Corollary
2.1 of [7], every subgroup of $S$ is contained by some $\eta$-class of $S$. Thus the subgroups $G_{i,j}$ \((i \in I, j \in J)\) are the maximal subgroups of $S$ and so $e$ is the identity element of some maximal subgroup $G_{i_0,j_0}$. Then

$$G_{i_0,j_0} = eG_{i_0,j_0}e \subseteq eS_\alpha e = e(eS_\alpha e)e \subseteq eK_\alpha e \subseteq G_{i_0,j_0}K_\alpha G_{i_0,j_0} \subseteq G_{i_0,j_0},$$

because $e$ is in the ideal $K_\alpha$ of $S_\alpha$, and $K_\alpha$ is a rectangular band of the subgroups $G_{i,j}$ \((i \in I, j \in J)\). Thus $eS_\alpha e = G_{i_0,j_0} \in \mathbf{H}$. Consequently, $\eta$ is an \textbf{LH}-congruence. In the next, we show that $\eta$ is a maximal \textbf{LH}-congruence. Let $\xi$ be an arbitrary \textbf{LH}-congruence of $S$ with $\eta \subseteq \xi$. We show that $\eta = \xi$. Let $S_\alpha$ and $S_\beta$ \((\alpha, \beta \in Y)\) be arbitrary $\eta$-classes of $S$. Assume that there is a $\xi$-class $T$ of $S$ such that $S_\alpha, S_\beta \subseteq T$. Let $e$ and $f$ be arbitrary idempotent elements of $S_\alpha$ and $S_\beta$, respectively. As $\xi$ is an \textbf{LH}-congruence, $eTe$ and $fTf$ are groups with the identity element $e$ and $f$, respectively. As $efe$ is in the group $eTe$, there is an element $x \in eTe$ (the inverse of $efe$ in $eTe$) such that $x(efe) = e$ which means that $e \in SfS$, that is, $f$ divides $e$. We can prove, in a similar way, that $e$ divides $f$. Then, by the definition of $\eta$ (see above), the idempotent elements $e$ and $f$ are in the same $\eta$-class of $S$. Thus $\alpha = \beta$. Consequently $\eta = \xi$ which implies that $\eta$ is a maximal \textbf{LH}-congruence of $S$.

Conversely, assume that the least semilattice congruence $\eta$ of $S$ is a maximal \textbf{LH}-congruence. Let $G$ be an arbitrary subgroup of $S$. Use the notations of the above part of the proof. By Corollary 2.1 of [7], $G$ is contained by an $\eta$-class $S_\alpha$ \((\alpha \in Y)\) of $S$; $S_\alpha$ is an ideal extension of $K_\alpha$ by a nil semigroup, and so $G \subseteq K_\alpha$; by Result 2.1, $K_\alpha$ is a rectangular band $B = I \times J$ of its (maximal) subgroups $G_{i,j}$ \((i \in I, j \in J)\) and so $G \subseteq G_{i_0,j_0}$ for some $i_0 \in I$ and $j_0 \in J$. Let $e$ denote the identity element of $G_{i_0,j_0}$. Then, as above,

$$G_{i_0,j_0} = eK_\alpha e = eS_\alpha e \in \mathbf{H}.$$

Thus $G$ belongs to $\mathbf{H}$. The theorem is proved.

An element $a$ of a semigroup $S$ is called periodic, if there are positive integers $n \neq k$ such that $a^n = a^k$. It is known that if $a$ is a periodic element of a semigroup $S$ then $\langle a \rangle = \{a, a^2, \ldots, a^i, a^{i+1}, \ldots, a^{i+m-1}\}$ is the subsemigroup of $S$ generated by $a$, in which $K_a = \{a^i, \ldots, a^{i+m-1}\}$ is a subgroup of $S$; here $i$ and $m$ denote the index and the period of $a$, respectively. It is clear that an element $a$ of a semigroup $S$ is periodic if and only if there is a positive integer $n$ such that $a^n$ is an idempotent element of $S$. A semigroup $S$ is called a periodic semigroup if every element of $S$ is periodic.

\textbf{Theorem 3.2} Let $\mathbf{H}$ be a given variety of groups. In a periodic Putcha semigroup $S$ containing finite many idempotent elements, the subgroups of $S$ belong to $\mathbf{H}$ if and only if the least semilattice congruence of $S$ is a maximal \textbf{LH}-congruence of $S$. 

Proof. Let $S$ be a periodic Putcha semigroup in which $E(S)$ is finite. By Result 2.3, $S$ is a semilattice $Y$ of archimedean semigroups $S_{\alpha}$ ($\alpha \in Y$). Let $\alpha \in Y$ be an arbitrary element. As $S_{\alpha}$ is periodic, it contains at least one idempotent element. As $S_{\alpha}$ is also archimedean, Result 2.4 implies that $S_{\alpha}$ is an ideal extension of a simple semigroup $K_{\alpha}$ containing at least one idempotent element by a nil semigroup. As $E(S)$ is finite, the simple semigroup $K_{\alpha}$ contains an idempotent element which is minimal in the ordering of idempotents of $K_{\alpha}$. Thus $K_{\alpha}$ is a completely simple semigroup. Then $S$ is a semilattice $Y$ of semigroups $S_{\alpha}$ ($\alpha \in Y$) such that every subsemigroup $S_{\alpha}$ is an ideal extension of a completely simple semigroup by a nil semigroup. Thus our assertion follows from Theorem 3.1.

A semigroup $S$ is called a left [right] Putcha semigroup if, for every $x, y \in S$, the assumption $y \in xS^1 [y \in S^1 x]$ implies $y^m \in x^2S^1 [y^m \in S^1 x^2]$ for some positive integer $m$.

Result 3.3 (Lemma 2.1 of [6] or Lemma 2.1 of [7]) $S$ is a left [right] Putcha semigroup if and only if, for any $x, y \in S$ and for any positive integer $n$, there is a positive integer $m$ such that $(xy)^m \in x^nS^1 [(xy)^m \in S^1 y^n]$.

Result 3.4 (Theorem 2.6 of [6] or Theorem 2.4 of [7]) A semigroup is an archimedean left and right Putcha semigroup containing at least one idempotent element if and only if it is a retract ideal extension ([7]) of a completely simple semigroup by a nil semigroup.

Theorem 3.5 Let $H$ be a given variety of groups. In a periodic left and right Putcha semigroup $S$, the subgroups of $S$ belong to $H$ if and only if the least semilattice congruence of $S$ is a maximal $\text{LH}_{\alpha}$-congruence of $S$.

Proof. Let $S$ be a periodic left and right Putcha semigroup. By Lemma 2.2 of [7], $S$ is a Putcha semigroup, and so, by Result 2.3, it is a semilattice $Y$ of archimedean semigroups $S_{\alpha}$ ($\alpha \in Y$). Let $\alpha \in Y$ be an arbitrary element. We show that $S_{\alpha}$ is a left and right Putcha semigroup. Let $n$ be a positive integer and $x, y \in S_{\alpha}$ be arbitrary elements. As $S$ is a left Putcha semigroup, by Result 3.3, there is a positive integer $m$ and an element $z \in S$ such that $(xy)^m = x^nz$. Let $\beta$ be the element of $Y$ such that $z \in S_\beta$. It is clear that $\beta \alpha = \alpha \beta = \alpha$. Then $(xy)^{m+1} = x^nz(xy)$ and $z(xy) \in S_\beta S_{\alpha} \subseteq S_{\beta \alpha} = S_{\alpha}$. Thus $(xy)^{m+1} \in x^nS_{\alpha}$. Then, by Result 3.3, $S_{\alpha}$ is a left Putcha semigroup.

We can prove, in a similar way, that $S_{\alpha}$ is a right Putcha semigroup. As $S$ is a periodic semigroup, the left and right Putcha archimedean semigroup $S_{\alpha}$ contains an idempotent element. Then, by Result 3.4, $S_{\alpha}$ is an ideal extension of a completely simple semigroup by a nil semigroup. Thus $S$ is a semilattice $Y$ of the semigroups $S_{\alpha}$ such that, for every $\alpha \in Y$, the subsemigroup $S_{\alpha}$ is...
an ideal extension of a completely simple semigroup by a nil semigroup. Thus our assertion follows from Theorem 3.1.

A semigroup $S$ is called a permutative semigroup if there is a positive integer $n$ and a non-identity permutation of $\{1, \ldots, n\}$ such that $S$ satisfies the identity $x_1 \ldots x_n = x_{\sigma(1)} \ldots x_{\sigma(n)}$.

**Result 3.6** (Corollary 1.4 of [9]) Every permutative semigroup is a semilattice of permutative archimedean semigroups.

**Result 3.7** (Theorem 2 of [8]) Every permutative archimedean semigroup containing at least one idempotent element is an ideal extension of a rectangular abelian group by a nil semigroup.

**Theorem 3.8** Let $H$ be a given variety of groups. In a periodic permutative semigroup $S$, the subgroups of $S$ belong to $H$ if and only if the least semilattice congruence of $S$ is a maximal $LH$-congruence of $S$.

**Proof.** Let $S$ be a periodic permutative semigroup. By Result 3.6, $S$ is a semilattice $Y$ of permutative archimedean semigroups $S_\alpha$ ($\alpha \in Y$); the corresponding congruence $\eta$ is the least semilattice congruence of $S$ (see Result 2.3). As $S$ is periodic, every $S_\alpha$ contains at least one idempotent element. Then, by Result 3.7, every $S_\alpha$ is an ideal extension of a rectangular abelian group by a nil semigroup. Then our assertion follows from Theorem 3.1.

### 4 Corollaries

In this section, we present some corollaries on the connection between the Rhodes radical and the least semilattice congruence of finite Putcha semigroups.

Let $K$ be a field and $S$ a finite semigroup. The semigroup algebra of $S$ over $K$ is denoted by $KS$. Recall that this is a $K$-vector space with basis $S$ and the multiplication extending the multiplication in $S$. As the basis of $KS$ is finite, the semigroup algebra $KS$ has a largest nilpotent ideal. This will be denoted by $\text{Rad}(S)$, and is called the Jacobson radical of $KS$.

Consider the composite mapping $K \mapsto KS \mapsto KS/\text{Rad}(KS)$; this is a morphism of semigroups where the latter two are viewed with respect to their multiplicative structure. The associated congruence $\text{Rad}_K(S)$ on $S$ is called the Rhodes radical of $S$.

For an arbitrary field $K$, let $G_K$ denote the variety $I$ or $G_p$ depending on the characteristic of $K$, where $I$ is the variety consisting of only the trivial semigroup, and $G_p$ is the variety of $p$-groups ($p$ is a prime). Let $G_K = I$ if $\text{char}K = 0$, and let $G_K = G_p$ if $\text{char}K = p$. 
Result 4.1 (Theorem 3.6 of [2]) The Rhodes radical of a finite semigroup $S$ over a field $K$ is the largest $LG_K$-congruence on $S$.

Corollary 4.2 The Rhodes radical of a finite Putcha semigroup $S$ over a field $K$ equals the least semilattice congruence of $S$ if and only if every subgroup of $S$ belongs to the variety $G_K$.

Proof By Result 4.1 and Theorem 3.2, it is obvious.

A semigroup is called a trivial semigroup if it contains only one element.

Corollary 4.3 The least semilattice congruence of a finite Putcha semigroup equals the Rhodes radical of $S$ over an arbitrary field if and only if every subgroup of $S$ is trivial.

Proof. By Corollary 4.2, it is evident.

Corollary 4.4 In an arbitrary finite band, the least semilattice congruence equals the Rhodes radical of $S$ over an arbitrary field.

Proof. As every band is a Putcha semigroup, the assertion follows from Corollary 4.3.

Corollary 4.5 The least semilattice congruence of a finite permutative semigroup $S$ equals the Rhodes radical of $S$ over an arbitrary field if and only if $S$ is a semilattice of semigroups which are ideal extension of a rectangular band by a nil semigroup.

Proof. As every permutative semigroup $S$ is a semilattice of permutative archimedean semigroups (see Result 3.6) and every finite permutative archimedean semigroup is an ideal extension of a rectangular abelian group by a nil semigroup (see Result 3.7), our assertion follows from Theorem 3.5, Result 4.1 and Corollary 4.3.

Corollary 4.6 The least semilattice congruence of a finite commutative semigroup $S$ equals the Rhodes radical of $S$ over every field if and only if $S$ is a semilattice of commutative nil semigroups.

Proof. As a commutative semigroup is permutative and a rectangular band is commutative if and only if it is trivial, our assertion follows from Corollary 4.5.
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