Upper Bound for the Hilbert Coefficients of Almost Cohen-Macaulay Algebras

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Abstract

Let $S$ be an almost Cohen-Macaulay Algebra over a field $K$, with a quasi-pure resolution. We establish an upper bound for the Hilbert Coefficients of $S$ in terms of the shifts in the minimal free resolution.

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1 Introduction

Let $K$ be a field and $S$ a homogeneous $K$-algebra. In other words, $S$ is a finitely generated graded $K$-algebra, generated over $K$ by elements of degree one. $S$ is isomorphic to $R/I$, where $R = K[x_1, x_2, \ldots, x_n]$ is the polynomial ring in $n$ variables and $I$ is a graded ideal contained in $(x_1, \ldots, x_n)$.

Let $H(S,i) = \dim_K S_i$ be the Hilbert function of $S$, and $d$ the dimension of $S$. It was shown that $H(S,i)$, for $i >> 0$, is a polynomial of degree $d-1$, see [[3],4.1.3]. This polynomial is called the Hilbert polynomial of $S$ and is written of the form

$$P_S(X) = \sum_{i=0}^{d-1} (-1)^{d-1-i} e_{d-1-i} \left( X + i \right)$$

Then the coefficients $e_{d-1-i}$ are called the Hilbert coefficients of $S$. In particular $e_0$ is called the multiplicity of $S$ and denoted by $e$.

Let $F$ be the minimal homogeneous resolution of $S = R/I$ over $R$ given by

$$0 \rightarrow \bigoplus_{j=1}^{b_1} R(-d_{i1}) \xrightarrow{\delta_1} \cdots \rightarrow \bigoplus_{j=1}^{b_i} R(-d_{ij}) \xrightarrow{\delta_{i-1}} \cdots \rightarrow \bigoplus_{j=1}^{b_{d-1}} R(-d_{ij}) \xrightarrow{\delta_d} R \rightarrow R/I \rightarrow 0$$
In 1993, Huneke and Srinivasan conjectured that if \( S \) is Cohen-Macaulay (\( \text{codim} \ S = \text{pdim} \ S = s \)), then the multiplicity satisfies the following lower and upper bounds

\[
\prod_{i=1}^{s} m_i \leq s! e(S) \leq \prod_{i=1}^{s} M_i
\]

where \( m_i = \min_j d_{ij} \) and \( M_i = \max_j d_{ij} \) for \( 1 \leq i \leq s \).

In 1998, Herzog and Srinivasan [5] conjectured that in the non Cohen-Macaulay case, if \( S \) is of codimension \( h \) then the multiplicity satisfies the upper bound

\[
h! e(S) \leq \prod_{i=1}^{h} M_i
\]

They also proved both conjectures in many different cases. In 2003, Boij and Soderberg [1] made stronger and natural conjectures which would imply the multiplicity conjecture. In 2008, these conjectures were proved by Eisenbud and Schreyer [4] for C-M modules in characteristic zero and extended to non C-M modules by Boij and Soderberg [2].

In 2008 also, Herzog and Zheng [6] used the above results to show that if \( S \) is Cohen-Macaulay of codimension \( s \) then all Hilbert coefficients satisfy

\[
\frac{m_1 m_2 \ldots m_s}{(s+i)!} h_i(m_1, \ldots, m_s) \leq e_i(S) \leq \frac{M_1 M_2 \ldots M_s}{(s+i)!} h_i(M_1, \ldots, M_s)
\]

with \( m_i = \min_j d_{ij} \), \( M_i = \max_j d_{ij} \) for \( 1 \leq i \leq s \) and

\[h_i(d_1, \ldots, d_s) = \sum_{1 \leq j_1 \leq \ldots \leq j_s \leq s} \prod_{k=1}^{i} (d_{j_k} - (j_k + k - 1)) \] and \( h_0(d_1, \ldots, d_s) = 1 \)

In this paper, we give a generalization to all coefficients in the best possible non Cohen-Macaulay case namely when \( S \) is almost Cohen-Macaulay (\( \text{codim} \ S = s - 1 \)) with quasi-pure resolution. In the first section we give a formula of the \( e_i(S) \) when \( S \) is almost Cohen-Macaulay, and in the second section we use this formula to show,

**Theorem 4.2.** Suppose \( S \) is almost Cohen-Macaulay of codimension \( s - 1 \), then

\[
e_k(S) \leq \frac{M_1 M_2 \ldots M_{s-1}}{(s-1+k)!} h_k(M_1, \ldots, M_{s-1})
\]

where \( h_k(d_1, \ldots, d_{s-1}) = \sum_{1 \leq j_1 \leq \ldots \leq j_{s-1} \leq s-1} \prod_{i=1}^{k} (d_{j_i} - (j_i + i - 1)) \)

Our theorem is an extension of Herzog and Srinivasan’s theorem for the multiplicity in [[5], theorem 1.5].
2 Preliminaries

Let \( R = K[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables and \( S \) a homogeneous \( K \)-algebra, so \( S = R/I \) with \( I \) a graded ideal contained in \((x_1, x_2, \ldots, x_n)\). Let \( F \) be the minimal homogeneous resolution of \( S \) over \( R \) given by:

\[
0 \to \bigoplus_{j=1}^{b_s} R(-d_{s,j}) \xrightarrow{\delta_s} \cdots \to \bigoplus_{j=1}^{b_1} R(-d_{1,j}) \xrightarrow{\delta_1} R \to R/I \to 0
\]

We denote the minimal and maximal shifts in the resolution by \( m_i = \min_j d_{ij} \) and \( M_i = \max_j d_{ij} \) for \( 1 \leq j \leq s \).

**Definition 2.1** A resolution is called quasi-pure if \( d_{ij} \geq d_{i-1,l} \) for all \( i \) and \( l \), that is, if \( m_i \geq M_i \) for all \( i \).

**Definition 2.2** \( S \) is said to be almost Cohen-Macaulay if the codimension of \( S \) is one less than its projective dimension \( s \).

Given \((\alpha_1, \alpha_2, \ldots, \alpha_h)\) a sequence of real numbers, we denote the following Vandermonde determinants by

\[
V_r = V_r(\alpha_1, \alpha_2, \ldots, \alpha_h) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_h \\
\alpha_1^2 & \alpha_2^2 & \cdots & \alpha_h^2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{h-2} & \alpha_2^{h-2} & \cdots & \alpha_h^{h-2} \\
\alpha_1^{h+r-1} & \alpha_2^{h+r-1} & \cdots & \alpha_h^{h+r-1}
\end{vmatrix}
\]

\( V_r \) factors out as:

\[
\prod_{1 \leq j < i \leq h} (\alpha_i - \alpha_j) \cdot \sum_{\beta_1 + \beta_2 + \ldots + \beta_h = r} (\alpha_1^{\beta_1} \cdot \alpha_2^{\beta_2} \cdots \alpha_h^{\beta_h})
\]

**Remark 2.3** \( V_r(\alpha_1, \alpha_2, \ldots, \alpha_h) \geq 0 \) if the sequence is in ascending order.

3 Hilbert coefficients

The following lemma of Peskine and Szpiro [8] plays an important role in our theorems,

**Lemma 3.1** [Peskine-Szpiro] Let \( R, I \) and \( F \) be as above. Then if \( h \) is the height of \( I \),

\[
\sum_{i=1}^{s} (-1)^i \sum_{j=1}^{b_i} d_{ij}^k = \begin{cases} 
-1 & k = 0 \\
0 & 1 \leq k < h \\
(-1)^{h!}e & k = h
\end{cases}
\]
Proof. Since the complex $F$ is exact, the alternative sum of the ranks of the free modules must be zero, so we get \[ \sum_{i=1}^{s} (-1)^i b_i + 1 = 0. \]

Now the Hilbert function is additive on short exact sequences so

\[
H_{R/I}(t) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} H_R(-d_{ij})(t) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} t^{d_{ij}} H_R(t)
\]

Now $H_R(t) = \sum_k H F(R,k) t^k$ where the Hilbert Function $H F(R,k) = \text{dim}_K R_k$, which is the number of monomials of total degree $k$ in $n$ variables. This is nothing but \[ \binom{n-1+k}{n-1}. \]

So $H_R(t) = \sum_k \left( \frac{n-1+k}{n-1} \right) t^k$ which is \[ \frac{1}{(1-t)^n} \]
and the Hilbert Series of $R/I$ is given by:

\[
H_{R/I}(t) = \frac{\sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} t^{d_{ij}}}{(1-t)^n}
\]

On the other hand the Hilbert Series is known to be a rational function of the following form, see [3]

\[
\frac{Q(t)}{(1-t)^d}
\]

where $d = \text{dim } R/I$ and $Q(1) = e$.

Comparing these two formulas we obtain

\[
\sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} t^{d_{ij}} = Q(t)(1-t)^h
\]

(1)

We denote the sum on the left hand side by $S_{R/I}(t)$. We differentiate both sides $k$ times, and evaluate them at $t = 1$ to get

\[
S_{R/I}^{(k)}(1) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} \left( \frac{d_{ij}}{k} \right) k! = 0 \quad \text{for} \quad 1 < k < h
\]

\[ (-1)^h h! e \quad \text{for} \quad k = h \]

Using the fact that \[ \binom{d_{ij}}{k} k! \] is a polynomial of degree $k$ in $d_{ij}$ and hence the $d_{ij}$ can be written as sums of multiples of \[ \binom{d_{ij}}{r}, \quad r < k; \] we get the result.

In the proof of lemma 3.1, we denoted \[ \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} t^{d_{ij}} \] by $S_{R/I}(t)$ and it was shown that this sum is equal to $(1-t)^h Q(t)$. When both sides of equation
were differentiated $h$ times, we obtain a formula for the multiplicity $e$. In the following lemma, we continue differentiating equation (1), $(h+k)$ times to obtain similar formulas for the coefficients $e_k$ with $k \geq 1$.

**Lemma 3.2** Under the assumptions of lemma 3.1 we have:

\[
(-1)^h(h+k)!e_k = \sum_{r=0}^{k} (-1)^{k-r} a_{k-r}(h) \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} d_{ij}^{h+r}
\]

with $a_{k-r}(h) = \sum_{1 \leq b_1 < b_2 < \ldots < b_{k-r} \leq h+k-1} b_1 b_2 \ldots b_{k-r}$ and $a_0 = 1$.

**Proof.** We first start by differentiating $S_{R/I}(t) = (1-t)^h Q(t)$:

\[
S_{R/I}^{(h+k)}(t) = (-1)^h \binom{h+k}{h} h! Q^{(k)}(t) + (1-t) P(t)
\]

where $P(t)$ is a polynomial in $t$. Evaluating at $t = 1$:

\[
S_{R/I}^{(h+k)}(1) = (-1)^h \binom{h+k}{h} h! Q^{(k)}(1) + 0
\]

Since $Q^{(i)}(1) = i!e_i$ we get:

\[
S_{R/I}^{(h+k)}(1) = (-1)^h(h+k)!e_k
\]

On the other hand, $S_{R/I}(t) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} t d_{ij}$. So

\[
S_{R/I}^{(k)}(1) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} \binom{d_{ij}}{k} k!
\]

\[
= \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} k! \prod_{r=0}^{k-1} (d_{ij} - r)
\]

\[
= \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} \sum_{r=1}^{k} (-1)^{k-r} a_{k-r} d_{ij}^r
\]

with $a_{k-r} = \sum_{1 \leq b_1 < b_2 < \ldots < b_{k-r} < k-1} b_1 b_2 \ldots b_{k-r}$ and $a_0 = 1$.

\[
S_{R/I}^{(h+k)}(1) = \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{h+k} (-1)^{h+k-r} a_{h+k-r} d_{ij}^r
\]

\[
= \sum_{r=1}^{h+k} (-1)^{h+k-r} a_{h+k-r} \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} d_{ij}^r
\]
with \( a_{h+k-r} = \sum_{1 \leq b_1 < b_2 < \ldots < b_{h+k-r} \leq h+k-1} b_1.b_2 \ldots .b_{h+k-r} \) and \( a_0 = 1 \)

Note that \( \sum_{i=0}^s (-1)^i \sum_{j=1}^{b_i} d_{ij}^r = 0 \) when \( r < h \) and

\[
S_{R/I}^{(h+k)}(1) = \prod_{r=0}^h (-1)^{h+k-r} a_{h+k-r} \sum_{i=0}^s (-1)^i \sum_{j=1}^{b_i} d_{ij}^r
\]

and hence the result.

We deduce that when \( S \) is almost Cohen-Macaulay in which case the codimension \( h \) of \( S \) is equal to \( s - 1 \), then we have

\[
(-1)^{s-1}(s - 1 + k)!e_k = \sum_{r=0}^k (-1)^{k-r} a_{k-r} \sum_{i=0}^s (-1)^i \sum_{j=1}^{b_i} d_{ij}^{s-1+r}
\]

(2)

with \( a_{k-r} = \sum_{1 \leq b_1 < b_2 < \ldots < b_{k-r} \leq s+k-2} b_1.b_2 \ldots .b_{k-r} \) and \( a_0 = 1 \)

4 Upper bound for the coefficients

Before we prove our main theorem, we compute a formula for the Hilbert coefficients as a function of Vandermonde determinants. This formula plays a pivotal role in our main theorem.

Theorem 4.1 Suppose \( S \) is almost Cohen-Macaulay of codimension \( s - 1 \), then

\[
-(s - 1 + k)!e_k = \frac{\sum_{r=0}^k (-1)^{k-r} a_{k-r} (|M_r| - |X_r|)}{\prod_{1 \leq j_i \leq b_i \atop 1 \leq i \leq s-1} V_r(d_{1j_1}, d_{2j_2}, \ldots, d_{s-1j_{s-1}})}
\]

with \( |M_r| = \sum_{1 \leq j_i \leq b_i \atop 1 \leq i \leq s} V_r(d_{1j_1}, d_{2j_2}, \ldots, d_{s-1j_{s-1}}) \) and

\[
|X_r| = \prod_{1 \leq j_i \leq b_i \atop 1 \leq i \leq s-1} V_r(d_{1j_1}, d_{2j_2}, \ldots, d_{s-1j_{s-1}})
\]

Proof.

To prove this theorem, we consider the following determinant
Upper bound for the Hilbert coefficients

\[ M_r = \begin{pmatrix}
    \sum_{j=1}^{b_1} 1 & \sum_{j=1}^{b_2} 1 & \ldots & \sum_{j=1}^{b_{s-1}} 1 & \sum_{j=1}^{b_s} 1 \\
    \sum_{j=1}^{b_1} d_{1j} & \sum_{j=1}^{b_2} d_{2j} & \ldots & \sum_{j=1}^{b_{s-1}} d_{(s-1)j} & \sum_{j=1}^{b_s} d_{sj} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \sum_{j=1}^{b_1} d_{ij}^{s-2} & \sum_{j=1}^{b_2} d_{2j}^{s-2} & \ldots & \sum_{j=1}^{b_{s-1}} d_{(s-1)j}^{s-2} & \sum_{j=1}^{b_s} d_{sj}^{s-2} \\
    \sum_{j=1}^{b_1} d_{ij}^{s-1+r} & \sum_{j=1}^{b_2} d_{2j}^{s-1+r} & \ldots & \sum_{j=1}^{b_{s-1}} d_{(s-1)j}^{s-1+r} & \sum_{j=1}^{b_s} d_{sj}^{s-1+r}
\end{pmatrix}
\]

of size \( s \) where \( s - 1 = \text{codim } S \).

We will compute the determinant of \( M_r \) in two different ways. First we replace the last column of \( M_r \) by the alternating sum of all columns of \( M_r \). The resulting matrix will be denoted by \( M_r' \). It is clear that

\[
|M_r| = (-1)^s |M_r'|.
\]

Moreover the last column of \( M_r \) becomes the transpose of the following vector

\[
(-1, 0, \ldots, s \sum_{i=0}^{s-1} (-1)^i \sum_{j=1}^{b_i} d_{ij}^{s-1+r}).
\]

So we have

\[
|M_r| = \sum_{i=0}^{s} (-1)^{s+i} \sum_{j=1}^{b_i} d_{ij}^{s-1+r} |W| + |X_r| \quad (3)
\]

where

\[
|W| = \det
\begin{pmatrix}
    \sum_{j=1}^{b_1} 1 & \sum_{j=1}^{b_2} 1 & \ldots & \sum_{j=1}^{b_{s-1}} 1 \\
    \sum_{j=1}^{b_1} d_{1j} & \sum_{j=1}^{b_2} d_{2j} & \ldots & \sum_{j=1}^{b_{s-1}} d_{(s-1)j} \\
    \vdots & \vdots & \ddots & \vdots \\
    \sum_{j=1}^{b_1} d_{ij}^{s-2} & \sum_{j=1}^{b_2} d_{2j}^{s-2} & \ldots & \sum_{j=1}^{b_{s-1}} d_{(s-1)j}^{s-2} \\
    \sum_{j=1}^{b_1} d_{ij}^{s-1+r} & \sum_{j=1}^{b_2} d_{2j}^{s-1+r} & \ldots & \sum_{j=1}^{b_{s-1}} d_{(s-1)j}^{s-1+r}
\end{pmatrix}
\]

\[
= \sum_{1 \leq j_1 \leq b_i \atop 1 \leq i \leq s-1} V(d_{1j_1}, d_{2j_2}, \ldots, d_{(s-1)j_{s-1}})
\]

and
So from equation 3, we have:

\[ (-1)^s \sum_{i=0}^{s} (-1)^i \sum_{j=1}^{b_i} d_{ij}^{s-1+r} = \frac{|M_r| - |X_r|}{|W|} \]

And from equation 2, we obtain

\[-(s - 1 + k)!e_k = \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} (|M_r| - |X_r|) \frac{1}{|W|} \]

On the other hand \(|M_r| = \sum_{1 \leq j_i \leq b_i, 1 \leq i \leq s} V_r(d_{1j_1}, d_{2j_2}, \ldots, d_{sj_s})\) and this gives us the result.

In fact, when the resolution is pure, i.e., there is only one shift at each step, the simplification is trivial. The C-M case was already studied in [6] and the almost C-M becomes

\[-(s - 1 + k)!e_k = \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} (|M_r| - |X_r|) \frac{1}{|W|} \]

Now we get to our main result.

**Theorem 4.2** Suppose \(S\) is almost Cohen-Macaulay of codimension \(s - 1\), then

\[ e_k(S) \leq \frac{M_1 M_2 \ldots M_{s-1}}{(s - 1 + k)!} h_k(M_1, \ldots, M_{s-1}) \]

where \(h_k(d_1, \ldots, d_s) = \sum_{1 \leq j_i \leq \cdots \leq s-1} \prod_{i=1}^{k} (d_{ji} - (j_i + i - 1))\).
Proof. In theorem 4.1 we showed that

\[-(s - 1 + k)!e_k \sum_{1 \leq j_i \leq b_i} V(d_1 j_1, \ldots, d_{s-1} j_{s-1}) + \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |X_r| =\]

\[\sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |M_r|\]

with

\[|M_r| = \sum_{1 \leq j_i \leq b_i} V_r(d_1 j_1, \ldots, d_s j_s)\]

\[= \sum_{1 \leq j_i \leq b_i} \sum_{1 \leq i \leq s} d_1^{\beta_1} \ldots d_s^{\beta_s} V(d_1 j_1, \ldots, d_s j_s)\]

So \[\sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |M_r| =\]

\[\sum_{1 \leq j_i \leq b_i} \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} \sum_{\beta_1 + \ldots + \beta_s = r} d_1^{\beta_1} \ldots d_s^{\beta_s} V(d_1 j_1, \ldots, d_s j_s)\]

We first want to show that \[\sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |M_r| \geq 0.\] Since the resolution is quasi-pure, \[V(d_1 j_1, \ldots, d_s j_s) \geq 0.\]

So it suffices to show that \[\sum_{r=0}^{k} (-1)^{k-r} a_{k-r} \sum_{\beta_1 + \ldots + \beta_s = r} d_1^{\beta_1} \ldots d_s^{\beta_s} \geq 0.\]

Let \[\alpha_i = d_i j_i\] for all \[i = 1 \ldots s\] and consider

\[\sum_{1 \leq l_1 \leq \ldots \leq l_k \leq s} \prod_{i=1}^{k} (\alpha_{l_i} - (l_i - 1 + i - 1))\]

Expanding the product, the above sum can be rearranged to be equal to

\[\sum_{r=0}^{k} (-1)^{k-r} \left( \sum_{0 \leq b_r < \ldots < b_{k-r} \leq s-1+k-1} b_1 b_2 \ldots b_{k-r} \sum_{l_1 \leq c_1 \leq \ldots \leq c_r \leq l_k \leq \ldots l_k \leq s} \alpha_{c_1} \ldots \alpha_{c_r} \right)\]

which is equal to

\[\sum_{r=0}^{k} (-1)^{k-r} \left( \sum_{0 \leq b_r < \ldots < b_{k-r} \leq s-1+k-1} b_1 b_2 \ldots b_{k-r} \sum_{\beta_1 + \ldots + \beta_k = r} \alpha_{\beta_1} \ldots \alpha_{\beta_k} \right)\]
which is the same as
\[ \sum_{r=0}^{k} (-1)^{k-r} \left( \sum_{1 \leq b_1 < \ldots < b_{k-r} \leq s-1+k-1} b_1 b_2 \ldots b_{k-r} \sum_{\beta_1 + \ldots + \beta_{k-r}} \alpha_{i_1}^{\beta_1} \ldots \alpha_{i_s}^{\beta_s} \right) \]

So \( \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} \sum_{\beta_1 + \ldots + \beta_{k-r}} d_{ij_1} \ldots d_{ij_s} \) factors out as
\[ \sum_{1 \leq l_1 \leq \ldots \leq l_k \leq s} \prod_{i=1}^{k} (\alpha_{l_i} - (l_i - 1 + i - 1)) \] with \( \alpha_i = d_{ij_i} \) for all \( i = 1 \ldots s \).

Suppose \( S \) has a quasi-pure resolution.

We may assume that \( \alpha_1 < \alpha_2 < \ldots < \alpha_s \) since this will not affect the Hilbert function. We claim that either \( \prod_{i=1}^{k} (\alpha_{l_i} - (l_i - 1 + i - 1)) = 0 \), or else \( (\alpha_{l_i} - (l_i - 1 + i - 1)) > 0 \) for \( i = 1, \ldots k \).

To prove the claim we follow the same argument in the case of pure resolution in [6]. Suppose \( \prod_{i=1}^{k} (\alpha_{l_i} - (l_i - 1 + i - 1)) \neq 0 \). Since \( \alpha_i \geq i \) for all \( i \), we must then have that \( \alpha_i - l_i + 1 > 0 \). Assume that not all factors \( \alpha_{l_i} - (l_i - 1 + i - 1) \) are positive and let \( m \) be the smallest positive integer with \( \alpha_{l_m} - (l_{m-1} + m - 1) < 0 \), then \( m > 1 \) and \( \alpha_{l_{m-1}} - (l_{m-1} - 1 + m - 2) > 0 \).

It follows that
\[ \alpha_{l_{m-1}} - (l_{m-1} - 1 + m - 2) - (\alpha_{l_m} - (l_{m-1} - 1 + m - 1)) \geq 2 \]

or equivalently
\[ l_m - l_{m-1} \geq \alpha_{l_m} - \alpha_{l_{m-1}} + 1 \]

This is a contradiction since \( \alpha_1 < \alpha_2 < \cdots \alpha_s \).

So equation (4) becomes
\[ -(s-1+k)!e_k \sum_{\substack{1 \leq j_1 \leq b_1 \\ 1 \leq i \leq s-1}} V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) + \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |X_r| \geq 0 \]

and
\[ (s-1+k)!e_k \sum_{\substack{1 \leq j_1 \leq b_1 \\ 1 \leq i \leq s-1}} V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \leq \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |X_r| \]
Again

\[ |X_r| = \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{s-1} d_{ij_i} V_r(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \]

\[ = \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{s-1} d_{ij_i} \sum_{\beta_1 + \ldots + \beta_{s-1} = r} d_{1j_1}^{\beta_1} \ldots d_{s-1j_{s-1}}^{\beta_{s-1}} V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \]

So

\[ \sum_{r=0}^{k} (-1)^{k-r} a_{k-r} |X_r| = \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{s-1} d_{ij_i} h_k(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \]

with

\[ h_k(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) = \sum_{1 \leq l_i \leq s-1} \prod_{i=1}^{k} (\alpha_{l_i} - (l_i + i - 1)) \text{ and } \alpha_{l_i} = d_{ij_i} \]

Since \( h_k(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \geq 0 \) and \( V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \geq 0 \) then,

\[ (s-1+k)! c_k \sum_{1 \leq j_i \leq b_i} \prod_{i=1}^{s-1} M_i h_k(M_1, \ldots, M_{s-1}) \sum_{1 \leq j_i \leq b_i} V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \leq \]

\[ \prod_{i=1}^{s-1} M_i h_k(M_1, \ldots, M_{s-1}) \sum_{1 \leq j_i \leq b_i} V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) \]

And since at least one of the \( V(d_{1j_1}, \ldots, d_{s-1j_{s-1}}) > 0 \), we obtain the result.

References


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