Asymptotic Behaviour and Cofiniteness of Generalized Local Cohomology Modules

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Abstract

In this paper, we prove that for any ideal $I$ of dimension one, $H^i_I(M,N)$ is $I$-cofinite for all $i$ and for any finite $R$-modules $M,N$. Furthermore, using the above result, in certain graded situations, the behaviour of the $n$-th graded component $H^i_{R^+}(M,N)_n$ of the generalized local cohomology modules with respect to irrelevant ideal $R^+$ as $n \to -\infty$ is studied.

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1 INTRODUCTION

Throughout this paper $R$ is a commutative Noetherian local ring. For unexplained terminology from homological and commutative algebra we refer to [5] and [6]. Generalized local cohomology was given in the local case by J.Herzog [12] and in the more general case by Bijan-Zadeh [1]. Let $R$ be a commutative Noetherian ring with identity, $I$ an ideal of $R$ and let $M,N$ be two $R$-modules. For an integer $i \geq 0$, the $i$-th generalized local cohomology module $H^i_I(M,N) = \lim \ Ext^i_R(\frac{M}{I^nM},N)$ with $M = R$, we obtain the ordinary local cohomology module $H^i_I(N)$ of $N$ with respect to $I$ which was introduced by Grothendieck. In [10] Grothendieck conjectured that for any ideal $I$ of $R$ and any finitely generated $R$-module $M$, the module $\text{Hom}(\frac{R}{I},H^i_I(M))$ is finitely generated, but soon Hartshorne was able to present a counterexample to Grothendieck conjecture (see[11] for details and proof). However, he
defined an $R$-module $T$ to be $I$-cofinite if $\text{Supp}(T) \subseteq V(I)$ and $\text{Ext}^i_R(R/I, T)$ is finitely generated for all $i \geq 0$.

In [11], Hartshorne proved that if $R$ is a complete regular local ring and $p$ is a prime ideal such that $\dim \frac{R}{p} = 1$ then $H^i_p(N)$ is $p$-cofinite for any finitely generated $R$-module $N$. This result was later extended to more general local rings and one-dimensional ideals $I$ by Huneke and Koh in [13], by Delfino in [8] and Delfino and Marley in [7]. Finally, Yoshida proved in [19] that the local cohomology modules $H^i_I(N)$ are $I$-cofinite for all finite $R$-modules $N$ and any ideal $I$ of $R$ with $\dim \frac{R}{I} = 1$.

One of the principal aims of this paper is to extend the result of Yoshida to the generalized local cohomology modules.

Divaani-Aazar and Sazeedeh [9] proved that if $R$ is a complete local ring and $I$ is a prime ideal with $\dim \frac{R}{I} = 1$ and $pd(M) < \infty$, then $H^i_I(M, N)$ is $I$-cofinite for any finitely generated $R$-modules $M, N$. In this paper we prove the above mentioned result without imposing the assumptions of completeness of the ring, and the property $I \in \text{Spec}(R)$. Also we show that $H^i_{\mathfrak{m}}(M, T)$ is Artinian and $I$-cofinite for all $i \geq 0$ whenever $M$ is finitely generated and $T$ is $I$-cofinite. Next, we recall some known properties of generalized local cohomology modules which we need in this note.

For any ideal $I$ of $R$ and two $R$-modules $M$ and $N$ the following statements hold:

i) $H^i_I(M, N) \cong \text{Ext}^i_R(M, N)$ if $N = \Gamma_I(N)$.

ii) If $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ is an exact sequence of $R$-modules and $R$-homomorphisms, then there is a long exact sequence

$$0 \longrightarrow H^0_I(M, N') \longrightarrow H^0_I(M, N) \longrightarrow H^0_I(M, N'') \longrightarrow H^1_I(M, N') \longrightarrow \ldots$$

iii) For each $R$-modules $M$ and $N$, there is a long exact sequence (called the Mayer-Vietoris sequence for $M, N$ with respect to $a$ and $b$)

$$0 \longrightarrow H^0_{a+b}(M, N) \longrightarrow H^0_a(M, N) \oplus H^0_b(M, N) \longrightarrow H^0_{a+b}(M, N) \longrightarrow H^1_{a+b}(M, N) \longrightarrow H^1_a(M, N) \oplus H^1_b(M, N) \longrightarrow \ldots$$

In the special case, we assume that $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is a positively graded Noetherian ring which is standard, in the sense $R = R_0[R_1]$ and set $R_+ = \bigoplus_{n \in \mathbb{N}} R_n$, the irrelevant graded ideal of $R$.

Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $N = \bigoplus_{n \in \mathbb{Z}} N_n$ be non-zero finitely generated graded $R$-modules. It is well known, that $H^i_{R_+}(M, N)$ carry a natural grading and that its grading has some similar properties as the ordinary graded local cohomology module $H^i_{R_+}(N)$.

Also, we will show that if $\dim R_0 \leq 1$, then the $\text{Ass}_{R_0} H^i_{R_+}(M, N)_n$ is asymptotically stable for $n \rightarrow -\infty$. The same theorem has been proved in [15,4.2]. Note that the statement of theorem (4.2) in [15] is true, but the
proof is not complete. In the proof of theorem [15,4.1], it is stated that
\( H^i_{R_0} (M, \frac{N}{xN}) \cong H^i_{(\frac{R_0}{xN})} (M, \frac{N}{xN}) \) which is no longer true. By using (3.3), it follows that \( \Gamma_{m_0 R_0} (H^i_{R_0+} (M, N)) \) are Artinian whenever \( q_0 \) is an \( m_0 \)-primary. The same result has been proved, with completely different arguments, in [3] for the case when \( M = R \).

Finally, we prove that if \( \dim R_0 = 2 \) and \( \Gamma_{m_0 R_0} (H^i_{R_0+} (M, N)) \) is Artinian, then \( \text{Ass}_{R_0} H^i_{R_0+} (M, N)_n \) is asymptotically stable for \( n \to -\infty \).

Brodmann et at.[2] proved the modules \( H^i_{R_0+} (N) \) are tame, if \( \dim R_0 = 2 \).

2 AUXILIARY TOOLS

Theorem 2.1 [cf ,19,1.1]. Let \((R,m)\) be a local Noetherian ring and \( I \) an ideal of \( R \) with \( \dim \frac{R}{I} = 1 \). Let \( M \) be a finite \( R \)-module. Then for any finite \( R \)-module \( N \) such that \( \text{Supp} N \subseteq V(I) \), we have \( \text{Ext}_R^i (N, H^i_I(M)) \) is of finite type for all \( i, j \).

Theorem 2.2 [cf 8, Th. 2]. Let \( R \) be a complete local ring and \( p \) a dimension one prime ideal of \( R \). Then the \( p \)-cofinite modules form an Abelian subcategory of the category of all \( R \)-modules.

Lemma 2.3 [cf 8, Lemma 2], [13, Lemma 4.2]. Let \( I \) be an ideal of a Noetherian ring \( R \). Let \( t \geq 0 \) be an integer. Then for an \( R \)-module \( T \), the following conditions are equivalent:

(i) \( \text{Ext}_R^i (\frac{R}{I}, T) \) is finitely generated for all \( i \leq t \).
(ii) \( \text{Ext}_R^i (N, T) \) is finitely generated for any finite \( R \)-module \( N \) such that support in \( V(I) \) and for all \( i \leq t \).
(iii) \( \text{Ext}_R^i (\frac{R}{I}, T) \) is finitely generated for all \( i \leq t \).

Lemma 2.4 [cf 14, Remark 3]. Let \( R \) be a local ring with maximal ideal \( m \). Let \( \hat{R} \) be the \( m \)-adic completion of \( R \), \( T \) an \( R \)-module and let \( a \) be an ideal of \( R \). Then the following statements hold:

(a) \( \dim \frac{R}{a} = \dim \frac{\hat{R}}{a\hat{R}} \).
(b) \( \text{Supp}_R T \subseteq V(a) \) if and only if \( \text{Supp}_R T \otimes_R \hat{R} \subseteq V(a\hat{R}) \).
(c) \( T \) is a finitely generated \( R \)-module if and only if \( T \otimes_R \hat{R} \) is a finitely generated \( R \)-module.
(d) If \( M \) is a finitely generated \( R \)-module, then
\[
H^i_a (M, N) \otimes_R \hat{R} \cong H^i_{a\hat{R}} (M \otimes_R \hat{R}, N \otimes_R \hat{R}).
\]
3 I-COFINITE R-MODULE $H^i_1(M, N)$

**Theorem 3.1** Let $p$ denote a prime ideal of the complete local ring $(R, m)$ with $\dim_R^p = 1$, and let $M, N$ be two finitely generated $R$-modules with $\text{pd} M = \ell < \infty$. Then $H^i_p(M, N)$ is $p$-cofinite for all $i \geq 0$.

**Proof:** We prove the theorem by induction on $\ell$. If $\ell = 0$, then $M$ is free. Hence, in view of 2.1 and 2.2, the $R$-module $H^i_p(M, N)$ is $p$-cofinite. Suppose, inductively that $\ell > 0$ and the result has been proved for $\ell - 1$. Now we consider the exact sequence $0 \to M' \to F \to M \to 0$ to obtain the exact sequence $H^i_{p-1}(M', N) \to H^i_p(M, N) \to H^i_p(F, N)$. One can use the above exact sequence together with the inductive hypothesis and (2.2) to complete the inductive step.

**Corollary 3.2** Let $R$ be a complete local ring and $I$ a radical ideal of $R$ such that $\dim_R^I = 1$. Let $M, N$ be two finitely generated $R$-modules. Then $\text{Ext}^i_R(R^I, H^i_1(M, N))$ is of finite type for all $i, j$.

**Theorem 3.3** Let $R$ be a local ring and let $I$ be an ideal of $R$ such that $\dim_R^I = 1$. Let $M, N$ be two finitely generated $R$-modules. Then $H^i_1(M, N)$ is $I$-cofinite for all $i \geq 0$.

**Proof:** As $\text{Supp} H^i_1(M, N) \subseteq V(I)$ by [9,3.2], it suffices to show that $\text{Ext}^i_R(R^I, H^i_1(M, N))$ is finitely generated for all $j, i$. By Lemma 2.3 and the fact that $H^i_1(M, N) \cong H^i_{\sqrt{I}}(M, N)$, we may assume that $I$ is a radical ideal of $R$ with $\dim_R^R = 1$. Also, in view of (2.4), we may suppose that $R$ is a complete local ring. The result follows from (3.2).

**Lemma 3.4** Let $\dim_R^I = 0$. Then $H^i_1(M, N)$ is Artinian and $I$-cofinite.

**Proof:** It follows immediately by using induction on $i$.

**Theorem 3.5** Let $R$ be a (not necessarily local) Noetherian ring. Let $N$ be an $I$-cofinite and $M$ be a finitely generated $R$-module. Then for every maximal ideal $m$ of $R$ and for all $t$, $H^t_m(M, N)$ is Artinian and $I$-cofinite.

**Proof:** By [18, Theorem 11.38], there is a Grothendieck spectral sequence $E_2^{p,q} := \text{Ext}^p_R(M, H^q_m(N)) \Rightarrow H^{p+q}_m(M, N) = E^n$. Since $N$ is $I$-cofinite, in view of [17, 1.4] and using induction on $q$, we see that $H^q_m(N)$ is Artinian and $I$-cofinite. It follows that $E_2^{p,q}$ is Artinian and $I$-cofinite for all $p, q$. By considering the sequence

$$\cdots \to E_2^{p-2,q+1} \to E_2^{p,q} \to E_2^{p+2,q-1} \to \cdots,$$

we deduce that $\text{im} d_2^{p-2,q+1}$ and $\ker d_2^{p,q}$ are Artinian and $I$-cofinite. Hence $E_3^{p,q} = \frac{\ker d_2^{p,q+1}}{\text{im} d_2^{p-2,q+1}}$ is Artinian and $I$-cofinite. By repeating these arguments
we get that \( E^{p,q}_r = \frac{\ker d^{p,q}_r}{\im d^{p,q}_{r-1}} \) is Artinian and I-cofinite for all \( r > 0 \) and hence \( E^{p,q}_\infty \) is Artinian and I-cofinite for all \( p,q \geq 0 \). There is a filtration
\[
E^n = E^n_0 \supseteq \ldots \supseteq E^n_p \supseteq \ldots \supseteq E^n_n \supseteq E^n_{n+1} = 0
\]
such that \( E^n_{p+1} \cong E^{p,n-p}_\infty \). Thus \( E^n_n \) is Artinian and I-cofinite.

Next, we consider the exact sequence \( 0 \to E^n_{p+1} \to E^n_p \to E^{p,n-p}_\infty \to 0 \)
for all \( p = 0,1, \ldots, n-1 \) to deduce that \( E^n_n \) is Artinian and I-cofinite.

\section{R_+-COFINITE AND ASYMPTOTIC BEHAVIOUR OF GRADED COMPONENTS}

In this section \( R = \bigoplus_{n \geq 0} R_n \) is a graded Noetherian ring with unique graded maximal ideal \( m \). Let \( R_+ = \bigoplus_{n > 0} R_n \) and \( M = \bigoplus_{n \in \mathbb{Z}} M_n \), \( N = \bigoplus_{n \in \mathbb{Z}} N_n \) be graded finitely generated \( R \)-modules.

In this section, we use our previous results to obtain conclusions on the behaviour of the \( R_0 \)-modules \( H^i_{R_+}(M,N)_n \) for \( n \ll 0 \).

The following notions are needed:

\begin{itemize}
  \item[A)] If \( T = \bigoplus_{n \in \mathbb{Z}} T_n \) is a graded \( R \)-module, we say that \( T \) is tame if there is some \( n_0 \in \mathbb{Z} \) such that either \( T_n = 0 \) for \( n \leq n_0 \) or \( T_n \neq 0 \) for \( n \leq n_0 \).

  Clearly all Artinian and all Noetherian \( R \)-modules are tame [cf 4].

  \item[B)] If \( (T_n)_{n \in \mathbb{Z}} \) is a family of \( R_0 \)-modules, we say that the set \( \Ass_{R_0}(T_n) \) is asymptotically stable for \( n \to -\infty \) if there is some \( n_0 \in \mathbb{Z} \) such that \( \Ass_{R_0}(T_n) = \Ass_{R_0}(T_{n_0}) \) for all \( n \leq n_0 \)[cf 2]. Asymptotic stability implies that \( H^i_{R_+}(M,N) \) is tame and \( \Ass_R H^i_{R_+}(M,N) \) is a finite set.
\end{itemize}

\textbf{Remark 4.1} Let \( R_0 \) be a Noetherian faithfully flat \( R_0 \)-algebra. Let \( R' = R_0 \otimes_{R_0} R \), \( M' = R_0 \otimes_{R_0} M = R' \otimes_R M \), and \( N' = R_0 \otimes_{R_0} N = R' \otimes_R N \). Then \( R' \) is a homogeneous Noetherian faithfully flat \( R \)-algebra, \( R'_+ = R_+ R' \) and \( M', N' \) are finitely generated graded \( R' \)-modules. Hence (by the graded flat base property) for each \( i \in \mathbb{N}_0 \) we have \( H^i_{R'_+}(M',N') \cong R' \otimes_R H^i_{R_+}(M,N) \), and so for each \( n \in \mathbb{Z} \), \( H^i_{R'_+}(M',N')_n \cong R'_0 \otimes_{R_0} H^i_{R_+}(M,N)_n \), as \( R_0 \)-modules. Therefore when this is the case \( H^i_{R'_+}(M',N')_n = 0 \) if and only if \( H^i_{R_+}(M,N)_n = 0 \). Also, if \( R_0 \) is local with unique maximal ideal \( m'_0 \), then \( \dim_R \frac{N}{m_0 N} = \dim_{R'} \frac{N'}{m'_0 N'} \) and, for an Artinian \( R \)-module \( T \), the \( R \)-module \( R'_0 \otimes_{R_0} T \) is Artinian.

\textbf{Theorem 4.2} Let \( R = R_0[R_1] \) with local base ring \( (R_0,m_0) \) and suppose that \( \frac{H^i_{R_+}(M,N)}{m_0 H^i_{R_+}(M,N)} \) is Artinian for all \( j \leq i \). Then \( \Ass_{R_0} H^i_{R_+}(M,N)_n \) is asymptotically stable for \( n \ll 0 \).

\textbf{Proof:} Let \( x \) be an indeterminate and apply 4.1 with \( R'_0 = R_0[x]_{m_0 R[x]} \). We
note \( R'_0 \) is Noetherian local faithfully flat \( R_0 \)-algebra with maximal ideal \( m_0R'_0 \) and with the residue filed isomorphic to \( (\frac{R_0}{m_0})(x) \) which is infinite. In view of \([16,23(ii)]\), we have \( \text{Ass}_{R_0}H^i_{R_+}(M,N)_n = \{ p'_0 \cap R_0 | p'_0 \in \text{Ass}_{R_0}H^i_{R_+}(M',N')_n \} \) for all \( n \in \mathbb{Z} \) and \( i \in \mathbb{N} \). Hence, if we replace \( R, M \) and \( N \), respectively, by \( R', M' \) and \( N' \), then we can assume that \( \frac{R_0}{m_0} \) is infinite. Now, we prove by induction on \( d = \dim \frac{N}{m_0N} \). The case \( d = 0 \), is clear as \( H^i_{R_+}(M,N) = \text{Ext}^i_R(M,\Gamma_{R_+}(N)) \). Now, suppose inductively, that \( d > 0 \) and that the result has been proved for \( d-1 \). In order to prove the inductive step, we may assume that \( \Gamma_{R_+}(N) = 0 \). Put \( A = \text{Ass}_{R_0}(N) \cup \bigcup_{j=1}^i \text{Att} \frac{H^i_{R_+}(M,N)}{m_0H^i_{R_+}(M,N)} - V(R_+) \). Since \( A \) is a finite set and \( \frac{R_0}{m_0} \) is infinite. Therefore, using \([6,1.5.12]\) together with \([3,3.1]\), indicates that there exists \( n_0 \in \mathbb{Z} \cup \{ \infty \} \) and an \( N \)-regular element \( x \in R_1 \) such that \( H^i_{R_+}(M,N)_n \xrightarrow{x} H^i_{R_+}(M,N)_{n+1} \) are epimorphism for \( j \leq i \) and all \( n < n_0 \). Using the exact sequence \( 0 \rightarrow N \xrightarrow{x} N \rightarrow \frac{N}{xN} \rightarrow 0 \) we obtain exact sequence \( 0 \rightarrow H^i_{R_+}(M,N)_{n+1} \rightarrow H^i_{R_+}(M,N)_n \xrightarrow{x} H^i_{R_+}(M,N)_{n+1} \) for all \( n \ll 0 \). Now, one can use the inductive hypothesis to see that \( \text{Ass}_{R_0}H^i_{R_+}(M,N)_n \) is asymptotically stable for \( n \rightarrow -\infty \).

Notation: Let \( i \in \mathbb{N}_0 \), and set

\[
A^i = \{ p \in \text{Ass}_R H^i_{R_+}(M,N) | \dim \frac{R_0}{p \cap R_0} \geq 1 \}
\]

\[
A^i_n = \{ p_0 \in \text{Ass}_{R_0} H^i_{R_+}(M,N)_n | \dim \frac{R_0}{p_0} \geq 1 \}.
\]

**Theorem 4.3** Let \( \dim R_0 \leq 1 \) and assume that \( R = R_0[R_1] \). Then \( \text{Ass}_{R_0} H^i_{R_+}(M,N)_n \) is asymptotically stable.

**Proof:** If \( \dim R_0 = 0 \), then the result follows by \((4.2)\) and \((3.4)\). So also assume that \( \dim R_0 = 1 \). Set \( \omega^i_n = \{ m_0 \} \cap \text{Ass}_{R_0} H^i_{R_+}(M,N)_n \) and observe that \( \text{Ass}_{R_0} H^i_{R_+}(M,N)_n = \omega^i_n \cup A^i_n \) for each \( n \in \mathbb{Z} \). By using \((3.3)\) and \([17,1.4]\) \( \text{Ass}_{R_0} H^i_{R_+}(M,N) \) is a finite set. Hence \( A^i \) and \( \bigcup_{n \in \mathbb{Z}} A^i_n \) are finite too. Now, let \( p_0 \in \bigcup_{n \in \mathbb{Z}} A^i_n \), then \( (R_0)_{p_0} \) is a local ring, \( \dim(\frac{R_0}{p_0})_{p_0} = 0 \) and hence \( \text{Ass}_{R_0} H^i_{R_+}(M,N)_{p_0} \) is asymptotically stable for \( n \rightarrow -\infty \). In view of the natural isomorphisms of \((R_0)_{p_0}\)-modules \( H^i_{R_+}(M,N)_{p_0} \cong H^i_{R_+}(M,N)_{p_0} \) for all \( n \ll 0 \) or \( p_0 \notin \text{Ass}_{R_0} H^i_{R_+}(M,N)_n \) for all \( n \ll 0 \). It follows that \( A^i_n \) is stable for \( n \rightarrow -\infty \). On the other hand in view of \([17,1.8]\) and \((3.3)\), it follows that \( \Gamma_{m_0R_0} H^i_{R_+}(M,N) \) is Artinian. Hence \( \Gamma_{m_0R_0} H^i_{R_+}(M,N) \) is tame. It follows either \( m_0 \in \text{Ass}_{R_0} H^i_{R_+}(M,N)_n \) for \( n \ll 0 \) or \( m_0 \notin \text{Ass}_{R_0} H^i_{R_+}(M,N)_n \) for \( n \ll 0 \). This proves our claim.

**Lemma 4.4** Suppose that \( (R_0,m_0) \) is local and \( \dim R_0 = 2 \). Then for all \( i \), \( \text{Ass}_{R_0} H^i_{R_+}(M,N) \) is a finite set.
Proof: Fix $i \in \mathbb{N}_0$. Let $x \in m_0 - \bigcup_{p \in \text{min}(R_0)} p$. Then $\dim(R_0)_x = 1$. Put $S = \{x^t | t \geq 0\}$. Then $S^{-1}H(R_0)_i(M, N) \cong H^i_{(R_0)_+}(M_x, N_x)$ is $(R_x)_+$-cofinite. Thus $\text{Ass}H^i_{(R_0)_+}(M_x, N_x)$ is a finite set \cite{17,1.4}.

We know $\text{Ass}S^{-1}H(R_0)_i(M, N) = \{S^{-1}p| p \in \text{Ass}H(R_0)_i(M, N), S \cap p = \emptyset\}$. Hence $\text{Ass}_RH(R_0)_i(M, N) \subseteq \{p|p \in \text{Ass}H(R_0)_i(M, N), x \notin p\} \cup \{p_0 + R_+|p_0 \in \text{Spec}(R_0), x \in p_0\}$. It follows that $\text{Ass}H(R_0)_i(M, N)$ is a finite set.

Corollary 4.5 Suppose that $\dim R_0 = 2$ and $\Gamma_{m_0}R_iH(R_0)_+ (M, N)$ is Artinian. Then $\text{Ass}R_0H(R_0)_+ (M, N)_n$ is asymptotically stable for $n \ll 0$.

Proof: $\text{Ass}H(R_0)_i(M, N)$, in view (4.4), is a finite set and the result follows with the same argument in (4.3).

References


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