On Almost Primary Ideals

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Abstract

In this paper a new type of ideals in commutative rings is defined which is called an almost primary ideal. Some properties of this type of ideals are obtained and also, some characterizations of them are given.

Keywords: almost prime, almost primary, weakly primary, localization

1 Introduction

Let \( R \) be a ring. If \( A, B \) are non-empty subsets of \( R \), the quotient \( A:B = \{ x \in R : xB \subseteq A \} \) \[4\]. If \( P \) is a proper ideal of \( R \), then it is called prime (resp. primary) if for \( a, b \in R \) such that \( ab \in P \), then \( a \in P \) or \( b \in P \) (resp. \( a \in P \) or \( b^n \in P \), for some positive integer \( n \)) \[8, page 168, 206\] and it is said to be weakly prime \[2\] (resp. weakly primary\[3\]) if for \( a, b \in R \) such that \( 0 \neq ab \in P \), then \( a \in P \) or \( b \in P \) (resp. \( a \in P \) or \( b^n \in P \), for some positive integer \( n \)) and it is called almost prime \[6\] if for \( a, b \in R \) such that \( ab \in P - P^2 \), then \( a \in P \) or \( b \in P \). If \( I \) is an ideal in \( R \), then \( I \) is called idempotent if \( I^2 = I \) \[10, page 9\] and it is called a finitely generated ideal if \( I = \langle a_1, ..., a_m \rangle \) for some \( a_i \in R, 1 \leq i \leq m \) \[5, page 182\]. The nil radical of \( I \) is denoted by \( \sqrt{I} \) and defined by \( \sqrt{I} = \{ x \in R : x^n \in I, n \in Z^+ \} \) \[7\]. \( R \) is said to be a fully idempotent ring if every ideal in \( R \) is idempotent \[10, page 14\], and it is called a Boolean ring if \( a^2 = a \) for all \( a \in R \) \[10, page 25\], and it is called a local ring if it has a unique maximal ideal \[5, page 226\]. A nonempty subset \( S \) of \( R \) is called a multiplicative set in \( R \) if \( 0 \notin S \) and \( a, b \in S \) implies \( ab \in S \) \[9, page 230\] and if \( R \) is commutative with identity, then the localization of \( R \) at \( S \) is denoted by \( R_S \). If \( A \) is an ideal of \( R \), then we define \( A^{-1} = \{ x \in R_S : xA \subseteq R \} \), where \( S \) is the set of all nonzero divisors of \( R \) and \( A \) is called an invertible ideal of \( R \) if \( AA^{-1} = R[1] \).

We will use the following statements in our results.
Lemma 1.1. [4] Let $P, A$ and $B$ be arbitrary ideals of a commutative ring $R$. If $P \subseteq A \cup B$, then $P \subseteq A$ or $P \subseteq B$. In particular, if $P = A \cup B$, then $P = A$ or $P = B$.

Lemma 1.2. Let $R$ be a commutative ring with identity and $S$ is a multiplicative set in $R$. Then $\langle a \rangle_S = \langle \frac{a}{m} \rangle$, for all $a \in R$ and $m \in S$.

Proof. Let $a \in R$ and $m \in S$. Let $\frac{x}{n} \in \langle a \rangle_S$, for $x \in \langle a \rangle$ and $n \in S$. Then $x = ra$, for some $r \in R$ and thus $\frac{x}{n} = \frac{ra}{n} = \frac{rm}{nm} \in \langle \frac{a}{m} \rangle$. Hence $\langle a \rangle_S \subseteq \langle \frac{a}{m} \rangle$. Next, let $\frac{x}{n} \in \langle \frac{a}{m} \rangle$, so there exists $\frac{r}{s} \in R_S$ such that $\frac{x}{n} = \frac{ra}{nm} = \frac{ra}{sm} \in \langle a \rangle_S$. Hence $\langle \frac{a}{m} \rangle \subseteq \langle a \rangle_S$ and thus $\langle a \rangle_S = \langle \frac{a}{m} \rangle$. 

Throughout this paper $R$ will denote a commutative ring with identity $1 \neq 0$ unless otherwise stated.

Now, we introduce the following definition.

Definition 1.3. We call a proper ideal $A$ of $R$ an almost primary ideal if for $a, b \in R$ such that $ab \in A - A^2$, then $a \in A$ or $b^n \in A$, for some positive integer $n$.

It is obvious that every primary ideal, every almost prime ideal and every idempotent ideal of $R$ is almost primary (and hence proper ideals of fully idempotent rings and of Boolean rings are almost primary), but the converse in all the above mentioned cases is not true in general. As an example for each case, we see that the ideal $(6)$ of $\mathbb{Z}_{30}$ is almost primary, but not primary and the second example is that in a Noetherian local ring with the unique non-zero maximal ideal $M$, we have $M^2$ is a primary ideal and hence almost primary but not almost prime [4]. The last example is that the ideal $(4)$ in $\mathbb{Z}_8$ is almost primary but not idempotent. Thus we can consider almost primary ideals as a generalization of primary, almost prime and idempotent ideals.

2 Main Results

Firstly, we give some elementary properties of almost primary ideals.

Proposition 2.1. If $I$ is an ideal of $R$, then the following statements are true.

(i) If $P$ is an almost primary ideal of $R$ with $I \subseteq P$, then $\frac{I}{P}$ is an almost primary ideal of $\frac{R}{P}$.
(ii) If $I$ is almost primary and $P'$ is a weakly primary ideal of $\frac{R}{I}$, then there exists an almost primary ideal $P$ of $R$ with $I \subseteq P$ such that $P' = \frac{P}{I}$.

Proof. (i) Let $(a + I)(b + I) \in \frac{P}{I} - (\frac{P}{I})^2$ and $(a + I) \notin \frac{P}{I}$. Then we get $ab \in P$, $ab + I \notin (\frac{P}{I})^2$ and $a \notin P$. Now, if $ab \notin P^2$, then for some $n \in Z^+$, we have $ab = \sum_{i=1}^{n} a_i b_i$ where $a_i, b_i \in P$ for all $i$ and thus we get $ab + I = \sum_{i=1}^{n} a_i b_i + I = \sum_{i=1}^{n} (a_i + I)(b_i + I) \in \frac{P}{I} \cap \frac{P}{I} = \frac{(\frac{P}{I})^2}{I}$ which is a contradiction, so that $ab \notin P^2$ and thus $b^n \in P$, for some $m \in Z^+$, which gives that $(b + I)^m \in \frac{P}{I}$. Hence $\frac{P}{I}$ is an almost primary ideal of $\frac{R}{I}$.

(ii) There exists an ideal $P$ of $R$ with $I \subseteq P$ such that $P' = \frac{P}{I}$. Let $a, b \in R$ be such that $ab \in P - P^2$. We have the following two cases:

Case (1) If $ab \in I$, then we get either $a \in I$ or $b^n \in I$ for some $n \in Z^+$, that gives either $a \in P$ or $b^n \in P$.

Case (2) If $ab \notin I$, then $(a + I)(b + I) \in \frac{P}{I} - 0$ and thus we get either $(a + I) \notin \frac{P}{I}$ or $(b + I)^n \in \frac{P}{I}$, for some $n \in Z^+$, which gives $a \in P$ or $b^n \in P$. Hence $P$ is almost primary.

Next, we prove that a homomorphic image of an almost primary ideal which contains the kernel is also an almost primary ideal.

**Proposition 2.2.** Let $f$ be a homomorphism from $R$ into the ring $R'$. If $P$ is an almost primary ideal of $R$ with $P \supseteq \ker f$, then $f(P)$ is an almost primary ideal of $f(R)$.

Proof. As $P$ is a proper ideal of $R$, one can get $f(P)$ is a proper ideal of $f(R)$. Now, let for $x, y \in f(R)$ we have $xy \in f(P) - (f(P))^2$, then $x = f(a)$ and $y = f(b)$ for some $a, b \in R$, so that $f(ab) = f(a)f(b) = xy \in f(P) - (f(P))^2$. Hence we have $f(ab) = f(p)$, for some $p \in P$. Thus we get $ab - p \in ker f \subseteq P$, which gives $ab \in P$. If $ab \in P^2$, then $xy = f(ab) \in f(P^2) = (f(P))^2$, which is a contradiction, so that $ab \notin P^2$. Hence either $a \in P$ or $b^n \in P$, for some $n \in Z^+$, which implies that $x \in f(P)$ or $y^n \in f(P)$.

Now, we give a property of an almost primary ideal, that concerning the zero divisors.

**Proposition 2.3.** Let $I$ be an almost primary ideal of $R$ and $b + I$ is a zero divisor in $\frac{R}{I}$. Then $b^n I \subseteq I^2$, for some $n \in Z^+$.

Proof. As $b + I$ is a zero divisor in $\frac{R}{I}$, there exists $c \notin I$ such that $bc \in I$. If we have, $b^n \in I$ for some $n \in Z^+$, then $b^n I \subseteq I^2$ and if $b^n \notin I$, for all $n \in Z^+$,
then we get $bc \in I^2$ (since $bc \notin I^2$ will give $c \in I$, and this is a contradiction). Further, for any $x \in I$, we have $x + c \notin I$ and $(x + c)b \in I$ and since $I$ is almost primary we must have $(x + c)b \in I^2$ and as $bc \in I^2$ we get $bx \in I^2$ which gives $bI \subseteq I^2$.

Next, we give some characterizations of almost primary ideals.

**Lemma 2.4.** Let $c$ be a nonzero nonunit in the ring of integers $Z$. If $p$ is a prime number such that $c \neq p^n$, for all $n \in Z^+$, then there exist $a, b \in Z$ for which $a \notin Zc, b^m \notin Zc$, for all $m \in Z^+$ and $ab \in Zc - Zc^2$.

**Proof.** As $Zc$ is not a primary ideal (since the primary ideals of $Z$ are 0 and $Z_p$ where $p \in Z^+$ and $p$ is a prime number), there exist $a, d \in Z$ such that $ad \in Zc$ with $a \notin Zc$ and $d^m \notin Zc$ for all $m \in Z^+$. If $ad \notin Zc^2$, then by putting $b = d$, we get $ab \in Zc - Zc^2$ and if $ad \in Zc^2$ then $d + c \notin Zc$ and $a(d + c) \in Zc$. If $a(d + c) \in Zc^2$ and as $ad \in Zc^2$, we get $ac \in Zc^2$. Hence $a \in Zc$, which is a contradiction. Thus we get $a(d + c) \notin Zc^2$, so take $b = d + c$ in this case.

Now, with the aid of the above lemma we prove that primary and almost primary ideals of the ring of integers with nonzero nonunit generators are equivalent.

**Proposition 2.5.** If $c$ is a nonzero nonunit element in $Z$, then $Zc$ is an almost primary if and only if it is primary.

**Proof.** The proof is immediate from Lemma 2.4.

Next, we prove that under certain conditions the result of Proposition 2.5, can be extended to any commutative ring with identity.

**Theorem 2.6.** Let $c$ be a nonzero nonunit element in a commutative ring with identity $R$. If $0 : \langle c \rangle \subseteq \langle c \rangle$, then $\langle c \rangle$ is an almost primary ideal of $R$ if and only if $\langle c \rangle$ if is a primary ideal of $R$.

**Proof.** ($\rightarrow$) Assume that $\langle c \rangle$ is almost primary but not primary. Then there exist $x, y \in R$ such that $xy \in \langle c \rangle$ but $x \notin \langle c \rangle$ and $y^n \notin \langle c \rangle$ for every $n \in Z^+$. Since $\langle c \rangle$ is almost primary we get $xy \in \langle c^2 \rangle$. Now we have $x(y + c) \in \langle c \rangle$. If $(y + c) \notin \langle c^2 \rangle$, then we have $x \in \langle c \rangle$ or $(y + c)^m \in \langle c \rangle$ for some $m \in Z^+$ that is, $x \in \langle c \rangle$ or $y^m \in \langle c \rangle$, which is a contradiction. Hence we have $x(y + c) \in \langle c^2 \rangle$, from which we get $xc \in \langle c^2 \rangle$, so that $xc = rc^2$ for some $r \in R$. Then $c(x - cr) = 0$ and as $0 : \langle c \rangle \subseteq \langle c \rangle$, we get $x - cr \in \langle c \rangle$ and hence $x \in \langle c \rangle$, which is a contradiction. Thus $\langle c \rangle$ must be a primary ideal of $R$.

The proof of the converse side is trivial.
Theorem 2.7. A proper ideal $I$ of $R$ is almost primary if and only if $I : x = I^2 : x$ or $I : x \subseteq \sqrt{I}$, for all $x \in R - I$.

Proof. Let $I$ be an almost primary ideal of $R$ and $x \in R - I$. Let $y \in I : x$, then $xy \in I$. If $xy \in I^2$, then $y \in I^2 : x$ and thus $I : x \subseteq I^2 : x$, from which we get $I : x = I^2 : x$ and if $xy \notin I^2$ and as $x \notin I$, we get $y^n \in I$ for some $n \in \mathbb{Z}^+$ and hence $y \in \sqrt{I}$, which means that $I : x \subseteq \sqrt{I}$.

Conversely, Let $xy \in I - I^2$ and $x \notin I$, then we get $y \in I : x$ and thus we get $y \in I^2 : x$ or $y \in \sqrt{I}$. But $y \in I^2 : x$ contradicts the fact that $xy \notin I^2$ and hence $y \in \sqrt{I}$, which means that $y^n \in I$, for some $n \in \mathbb{Z}^+$. \qed

Theorem 2.8. If $P$ is a proper ideal of $R$, then the following assertions are equivalent.

(i) $P$ is an almost primary ideal of $R$.

(ii) For $a \in R - \sqrt{P}$, we have $P : a = P \cup (P^2 : a)$.

(iii) For $a \in R - \sqrt{P}$, we have $P : a = P$ or $P : a = P^2 : a$.

Proof. (i)$\rightarrow$ (ii) Let $a \in R - \sqrt{P}$. Clearly, $P \cup (P^2 : a) \subseteq P : a$. Suppose that $b \in P : a$, so $ab \in P$. If $ab \in P^2$, then $b \in P^2 : a$ and if $ab \notin P^2$, then $b \in P$. Hence, we get $P : a = P \cup (P^2 : a)$.

(ii)$\rightarrow$ (iii) Let $a \in R - \sqrt{P}$ so that $P : a = P \cup (P^2 : a)$. Since $P, P : a$ and $P^2 : a$ are all ideals of $R$ so by Lemma 1.1, we get $P : a = P$ or $P : a = P^2 : a$.

(iii)$\rightarrow$ (i) Let $ab \in P - P^2$ with $b \notin \sqrt{P}$ that is, $b^n \notin P$, for all $n \in \mathbb{Z}^+$, then $a \in P : b$. If $P : b = P^2 : b$, we get $ab \in P^2$, which is a contradiction. Hence $P : b = P$, this means $a \in P$. \qed

Next, we prove the following lemma which leads to another characterization of almost primary ideals.

Lemma 2.9. Let $\{\mathfrak{A}_i : 1 \leq i \leq m\}$ be a finite set of left ideals in $R$ (not necessarily commutative). If some power of each $\mathfrak{A}_i$ is contained in an ideal $I$ of $R$, then some power of $\mathfrak{A}_1 + \mathfrak{A}_2 + \ldots + \mathfrak{A}_m$ is also contained in $I$.

Proof. To prove this lemma we use the principal of mathematical induction, it is enough to handle the case $m = 2$. Let some powers of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be contained in $I$, then we get $\mathfrak{A}_1^n \subseteq I$ and $\mathfrak{A}_2^n \subseteq I$ (where $n$ is the maximum of the powers of $\mathfrak{A}_1$ and $\mathfrak{A}_2$), we claim that $(\mathfrak{A}_1 + \mathfrak{A}_2)^{2n} \subseteq I$. Consider the product $(a_1 + b_1)(a_2 + b_2)...(a_{2n} + b_{2n})$, of $2n$ elements in $\mathfrak{A}_1 + \mathfrak{A}_2$ [where $a_i \in \mathfrak{A}_1, b_i \in \mathfrak{A}_2$]. When this product is expanded, each term of it is a product of $2n$ elements, some of them from $\mathfrak{A}_1$ and the others from $\mathfrak{A}_2$. In each of these
Theorem 2.10. Let $A$ and $B$ be two proper ideals of $R$ with $B$ is finitely generated. Then $I$ is almost primary if and only if $AB \subseteq I - I^2$ implies that $A \subseteq I$ or $B^n \subseteq I$ for some $n \in Z^+$.

Proof. Suppose that $I$ is almost primary. Since $B$ is finitely generated so there exists $k \in Z^+$ such that $B = \langle b_1 \rangle + \langle b_2 \rangle + \ldots + \langle b_k \rangle$, where $b_i \in B$ for every $1 \leq i \leq k$. If $A \nsubseteq I$, then there exists an $a \in A$ but $a \notin I$. Then for every $1 \leq i \leq k$, we have $ab_i \in AB \subseteq I - I^2$ and as $I$ is almost primary, there exists $l_i \in Z^+$ such that $b_i^{l_i} \in I$, from which we get $\langle b_i \rangle^{l_i} \subseteq I$ and by Lemma 2.9, we get $B^n \subseteq I$ for some $n \in Z^+$.

Conversely, suppose $a, b \in R$ with $ab \in I - I^2$. Then we have $\langle a \rangle \langle b \rangle \subseteq I - I^2$, so that $\langle a \rangle \subseteq I$ or $\langle b \rangle^n \subseteq I$ for some $n \in Z^+$ which gives that $a \in I$ or $b^n \in I$.

Remark 2.11. Consider the ideal $\langle 6 \rangle$ in $Z_{30}$. It is clear that $4 \times 9 \in \langle 6 \rangle^2$ but $4 \notin \langle 6 \rangle$ and $9 \notin \langle 6 \rangle$. That means there is an almost primary ideal which does not satisfy the condition (whenever $ab \in I^2$, for $a, b \in R$, then $a \in I$ or $b \in I$). Note that the ideal $\langle 4 \rangle$ of $Z_8$ is an almost primary ideal and satisfies the above condition but not prime. We call such an ideal which satisfies the above condition as a 2-potent prime ideal and now, we prove that in the set of all 2-potent prime ideals primary ideals and almost primary ideals are equivalent.

Proposition 2.12. If $I$ is a 2-potent prime ideal of $R$, then $I$ is almost primary if and only if it is primary.

Proof. Suppose $I$ is an almost primary ideal, and let $a, b \in R$ with $ab \in I$ but $a \notin I$. If $ab \in I^2$ then from the condition we get $b \in I$ and if $ab \notin I^2$ then as $I$ is almost primary, we get $b^n \in I$ for some $n \in Z^+$ and thus $I$ is a primary ideal. The proof of the converse side is trivial.

Proposition 2.13. Let $I$ be an ideal of $R$ with $\sqrt{I} = I$. Then $I$ is almost primary if and only if it is almost prime.

Proof. Suppose $I$ is almost primary. Let $a, b \in R$ with $ab \in I - I^2$. If $a \notin I$ then we get $b^n \in I$ for some $n \in Z^+$, and hence $b \in \sqrt{I} = I$ which means that $I$ is almost prime.

The proof of the converse side is trivial.

Next, we prove that invertible almost primary ideals are primary.
Theorem 2.14. If $I$ is an invertible ideal of $R$, then $I$ is almost primary if and only if it is a primary ideal.

Proof. Let $I$ be almost primary and $x, y \in R$ with $xy \in I$. If $x \notin I$ then $y + I$ is a zero divisor in $R/I$, so by Theorem 2.3, we get $y^n I \subseteq I^2$ for some $n \in \mathbb{Z}^+$, and as $I$ is an ideal, so we get $y^n R = y^n I^{-1} \subseteq I^2 I^{-1} = IR \subseteq I$, from which we get that $y^n \in I$. The proof of the converse side is trivial. □

Theorem 2.15. A proper ideal $I$ of $R$ is an almost primary ideal if and only if for any ideals $A, B$ of $R$ with $AB \subseteq I - I^2$ then $A \subseteq I$ or $B \subseteq \sqrt{I}$.

Proof. $(\rightarrow)$ Let $I$ be almost primary. If $B \subseteq \sqrt{I}$, then choose $b \in B$ such that $b \notin \sqrt{I}$. If $a$ is an arbitrary element of $A$, then $ab \in I - I^2$ and as $b \notin \sqrt{I}$, we get $a \in I$ so $A \subseteq I$.

$(\leftarrow)$ Let $ab \in I - I^2$. Then $(a) \langle b \rangle \subseteq I - I^2$ and hence $(a) \subseteq I$ or $(b) \subseteq \sqrt{I}$ which means $a \in I$ or $b^n \in I$, for some $n \in \mathbb{Z}^+$. □

Theorem 2.16. A proper ideal $I$ of $R$ is almost primary if and only if $\frac{I}{I^2}$ is a weakly primary ideal of $\frac{R}{I}$.

Proof. $(\rightarrow)$ Let $I$ be almost primary and $(a + I^2)(b + I^2) \in \frac{I}{I^2} - 0$, where $a + I^2, b + I^2 \in \frac{R}{I}$. Then $ab \in I$ and $ab \notin I^2$, but since $I$ is almost primary so either $a \in I$ or $b^n \in I$ for some $n \in \mathbb{Z}^+$. If $a \in I$ then $a + I^2 \in \frac{I}{I^2}$ and if $b^n \in I$ then $(b^n + I^2) = (b + I^2)^n \in \frac{I}{I^2}$.

$(\leftarrow)$ Let $\frac{I}{I^2}$ be a weakly primary ideal of $\frac{R}{I}$ and $ab \in I - I^2$ where $a, b \in R$, then $ab + I^2 \in \frac{I}{I^2}$ and $ab + I^2 \notin I^2$, from this we get $(a + I^2)(b + I^2) \in \frac{I}{I^2} - 0$, so either $(a + I^2) \in \frac{I}{I^2}$ or $(b + I^2)^n \in \frac{I}{I^2}$, for some $n \in \mathbb{Z}^+$, which gives either $a \in I$ or $b^n \in I$. □

Finally, we give some characterizations of almost primary ideals in certain types of rings.

It is well known that if $R$ is a commutative ring with identity and $S$ is a multiplicative set in $R$ and if $P$ is a prime ideal of $R$ with $P \cap S = \phi$, then $P_S$ is a prime ideal of $R_S$ and $P_S \cap R = P$ [4], where $P_S \cap R = \{x \in R; xs \in P$, for some $s \in S\}$. Now we extend the first part of the above result to almost primary ideals, but first we give the following example which shows that the second part of the above result may not hold in general. Consider the ring $Z_6$ and the multiplicative set $S = \{1, 2, 4\}$. If we take the almost primary ideal $P = \{0\}$ in $Z_6$, then we have $P_S \cap Z_6 = \{x \in Z_6 : xs = 0$, for some $x \in S\} = \{0, 3\} \neq P$.

Theorem 2.17. If $P$ is an almost primary ideal of $R$ and $S$ is a multiplicative set in $R$ with $P \cap S = \phi$, then $P_S$ is almost primary in $R_S$. 
Proof. If $P_S = R_S$, then take $s \in S$(since $S \neq \phi$), then $\frac{z}{s} \in P_S$. In fact $\frac{z}{s}$ is the identity element of $R_S$. Then $\frac{z}{s} = \frac{p}{t}$, for some $p \in P$ and $t \in S$. Thus, there exists $k \in S$ such that $kst = ksp \in P \cap S$ which is a contradiction and thus $P_S \neq R_S$. Now, let $\frac{pq}{st} \in P_S - (P_S)^2$ with $\frac{pq}{st} \notin P_S$, so that $\frac{pq}{st} \notin P_S, \frac{pq}{st} \notin (P_S)^2$ and $p \notin P$. Then $\frac{pq}{st} = \frac{z}{u}$, for some $z \in P$ and $u \in S$, so there exists $v \in S$ such that $\frac{v}{u} \notin P_S$ and if $\frac{pq}{st} \in P^2$, then we have $\frac{pq}{st} = \frac{uv}{st} = \frac{v}{u} \in (P_S)^2$, which is a contradiction and thus $\frac{pq}{st} \notin P^2$ so that $uv \notin P - P^2$. Now, if $\frac{pv}{u} \in P$, then $\frac{pv}{u} \in P - P^2$, so we get $(\frac{pv}{u})^n \in P$, for some $m \in \mathbb{Z}^+$ which is a contradiction as $P \cap S = \phi$. So that $pv \notin P$, which in consequence gives that $q^n \in P$, for some $n \in \mathbb{Z}^+$. Hence we get $(\frac{q}{s})^n = \frac{q^n}{s^n} \in P_S$. \hfill \Box

Next, we prove that the almost primariness property of principal ideals in local rings is preserved under localization at the unique maximal ideal.

**Theorem 2.18.** Let $R$ be a local ring with the unique maximal ideal $M$. If $a \in R$, then $\langle a \rangle$ is almost primary in $R$ if and only if $\langle a \rangle_M$ is almost primary in $R_M$.

*Proof.* $(\rightarrow)$ Let $\langle a \rangle$ be almost primary. As $\langle a \rangle$ is proper in $R$, we have $\langle a \rangle \subseteq M$, so $\langle a \rangle \cap (R - M) = \phi$, and hence by Theorem 2.17, $\langle a \rangle_M$ is almost primary in $R_M$.

$(\leftarrow)$ Let $\langle a \rangle_M$ be an almost primary ideal in $R_M$, and $x, y \in R$ be such that $xy \in \langle a \rangle - \langle a \rangle^2 = \langle a \rangle - \langle a^2 \rangle$. We have $\frac{x}{1} = \frac{y}{1} \in \langle a \rangle_M$. If $\frac{x}{1} = \frac{y}{1} \in \langle a^2 \rangle_M$, then we get $\frac{xy}{1} = \frac{z}{u}$, for some $z \in \langle a^2 \rangle$ and $u \notin M$. Hence there exists $u \notin M$ such that $uvxy = uz \in \langle a^2 \rangle$ and as $uv \notin M$, it is a unit in $R$. Hence we have $xy = (uv)^{-1}(uvxy) \in \langle a^2 \rangle$, which is a contradiction. Thus $\frac{x}{1} = \frac{y}{1} \notin \langle a^2 \rangle_M$ and as $\langle a \rangle_M$ is almost primary, we get either $\frac{x}{1} \not\in \langle a \rangle_M$ or $\frac{x}{1} = (\frac{z}{u})^n \in \langle a \rangle_M$ for some $n \in \mathbb{Z}^+$. If $\frac{z}{u} \in \langle a \rangle_M$, we get $x \in \langle a \rangle$ and if $\frac{z}{u} \in \langle a \rangle_M$ we get $y^n \in \langle a \rangle$ and thus $\langle a \rangle$ is almost primary in $R$. \hfill \Box

**Corollary 2.19.** Let $R$ be a local ring with the unique maximal ideal $M$. If $a \in R$, then $\langle a \rangle$ is almost primary in $R$ if and only if $\langle a \rangle_m$ is almost primary in $R_m$, for every $m \notin M$.

*Proof.* $(\rightarrow)$ Let $\langle a \rangle$ be almost primary in $R$ and $m \notin M$, then by Theorem 2.18 we get $\langle a \rangle_M$ is almost primary in $R_M$ and by Lemma 1.2, we have $\langle a \rangle_M = \langle \frac{a}{m} \rangle$ and thus $\langle \frac{a}{m} \rangle$ is almost primary in $R_M$.

$(\leftarrow)$ Let $\langle \frac{a}{m} \rangle$ be almost primary in $R_M$ for all $m \notin M$. So $\langle \frac{a}{m} \rangle$ is almost primary in $R_M$. By Lemma 1.2, we have $\langle a \rangle_M = \langle \frac{a}{m} \rangle$, so $\langle a \rangle_M$ is almost primary in $R_M$, then by Theorem 2.18, we get $\langle a \rangle$ is almost primary in $R$. \hfill \Box

Now we describe almost primary ideals in the direct product of any two given rings.
**Theorem 2.20.** Let $R$ and $S$ be two commutative rings with identity.

1. A proper ideal $A$ of $R$ is almost primary if and only if $A \times S$ is an almost primary ideal in $R \times S$.

2. A proper ideal $B$ of $S$ is almost primary if and only if $R \times B$ is an almost primary ideal in $R \times S$.

**Proof.** (1)$(\rightarrow)$ Let $A$ be an almost primary ideal in $R$. As $A$ is proper in $R$, we get $A \times S$ is a proper ideal in $R \times S$. Now let $(a, s), (b, t) \in R \times S$ be such that $(a, s), (b, t) \in A \times S - (A \times S)^2$ and $(a, s) \notin A \times S$, where $a, b \in R$ and $s, t \in S$. Then we get $(ab, st) \in A \times S - (A \times S)^2 = (A - A^2) \times S$ and $a \notin A$, which in consequence implies that $ab \in A - A^2$. Hence we get $b^n \in A$ for some $n \in \mathbb{Z}^+$, and this result gives that $(b, t)^n = (b^n, t^n) \in A \times S$.

$(\leftarrow)$ Let $A \times S$ be an almost primary ideal in $R \times S$ and $a, b \in R$ be such that $ab \in A - A^2$ with $a \notin A$, then we have $(a, 1_s)(b, 1_s) = (ab, 1_s) \in (A - A^2) \times S = A \times S - (A \times S)^2$. As $A \times S$ is almost primary, we get $(b, 1_s)^n \in A \times S$ for some $n \in \mathbb{Z}^+$, then $(b^n, 1_s) = (b, 1_s)^n \in A \times S$, which gives $b^n \in A$.

(2) The proof can be done as the same technique as in the above.

**Corollary 2.21.** Let $R$ and $S$ be two commutative rings with identity, then an ideal $K$ of $R \times S$ is almost primary if and only if it has one of the following forms:

(i) $I \times S$, where $I$ is an almost primary ideal of $R$.

(ii) $R \times J$, where $J$ is an almost primary ideal of $S$.

(iii) $I \times J$, where $I$ is an idempotent ideal of $R$ and $J$ is an idempotent ideal of $S$.

**Proof.** The proof will follow from Theorem 2.20 and the fact that every ideal $K$ of $R \times S$ is of the form $K = I \times J$, where $I$ is an ideal of $R$ and $J$ is an ideal of $S$.

**References**


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