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Some New Results on Fundamental Topological Algebras

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Abstract. Let $\mathcal{A}$ ba a fundamental topological algebra. In this article we study the fundamentality of $\theta(\mathcal{A})$ for which θ is a one to one uniformly continuous homomorphism. Nonvoidness of spectrum of some elements of an FLM algebra and its carrier space is also studied.

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1 Introduction

The fundamental topological spaces (also algebras) has been introduced in [1] in 1990 extending the meaning of both local convexity and local boundedness. The most famous Cohen factorization theorem has been extended and proved for complete metrizable fundamental topological algebras in [1]. Some basic theorems as well as are proved on fundamental topological vector spaces and fundamental topological algebras in [2], [3], [4] and [5]. A natural question is to ask for generalizing the basic results on these new class of topological algebras.

The fundamental locally multiplicative topological algebras (abbreviated by FLM) with a property very similar to the normed algebras is also introduced in [2]. Also in [2] a topological structure is defined on the algebraic dual space of an FLM algebra to make it a normed space, and some of the famous theorems of Banach algebras are extended for complete metrizable FLM algebras. Here, in this note we investigate the fundamentality of image of a fundamental topological algebra under a one to one uniformly continuous homomorphism with dense range. In sequel we study the relation between maximal ideal spaces of an FLM topological algebra with its ideals.

2 Definitions and Preliminaries

Definition 2.1. A topological linear space $\mathcal{A}$ is said to be fundamental one if there exists $b > 1$ such that for every sequence (x_n) of $\mathcal{A}$, the convergence of $b^n(x_n - x_{n-1})$ to zero in $\mathcal{A}$ implies that (x_n) is Cauchy.

Definition 2.2. A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

Definition 2.3. A fundamental topological algebra is called to be locally multiplicative , if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero ,the sufficiently large powers of U_0 lie in V. These kind algebras are called FLM in abbreviation.

Let $\mathcal{A}$ be a topological algebra with identity, by spectrum of an element $x \in \mathcal{A}$ we mean $\{\lambda \in \mathbb{C} : \lambda e - x \ \ is \ \ not \ \ invertible\}$ and denote it by $sp(x)$, if $\mathcal{A}$ has no identity we define

$$sp(x) = \{\lambda \in \mathbb{C}\backslash\{0\} : \lambda^{-1}x \ \ is \ \ not \ \ quasi \ \ invertible\}.$$

3 Fundamental Topological Algebras

In this section we would like to study that under which conditions the fundamentality of an algebra can be transferred into another one. We will need the following lemma in next theorem, which its proof is omitted because of clearness.

Lemma 3.1. *Let $\mathcal{A}$ be a fundamental topological algebra, then the closure of $\mathcal{A}$ is also a fundamental topological algebra.*

Theorem 3.2. *Let $\mathcal{A}$ be a fundamental topological algebra and $\mathcal{B}$ be a topological algebra. If $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ is a one to one uniformly continuous homomorphism with dense range in $\mathcal{B}$, then $\mathcal{B}$ is a fundamental topological algebra.*

Proof : Let $c^n(y_n - y_{n-1})$ tends to zero in $\mathcal{B}$, for each $c > 1$ and for some sequence (y_n) in $\mathcal{B}$. Fix n, then there exists sequence $(y_{n,m})$ in $\theta(\mathcal{A})$ such that $y_{n,m} \longrightarrow y_n$, as $m \longrightarrow \infty$. According to lemma 3.1, it is sufficient to show that $\theta(\mathcal{A})$ is a fundamental topological algebra.

Let for sequence (y_n) in $\theta(\mathcal{A})$, there exists $c > 1$ such that $c^n(y_n - y_{n-1}) \longrightarrow 0$. Fix n, then there exists $x_n \in \mathcal{A}$ such that $\theta(x_n) = y_n$. Then according to our assumption we have

$$\theta(c^n(x_n - x_{n-1})) = c^n(y_n - y_{n-1}) \longrightarrow 0,$$

therefore $c^n(x_n - x_{n-1}) \longrightarrow 0$. Since $\mathcal{A}$ is a fundamental, thus (x_n) is a Cauchy sequence and thereupon (y_n) is a Cauchy sequence.$\square$

4 New Results For FLM Algebras

In a Banach algebra $\mathcal{B}$ for every elements $x \in \mathcal{B}$, $sp(x)$ is nonvoid and compact [7], compactness of $sp(x)$ for complete metrizable FLM algebras with a unit element is proved in [2] (theorem 4.4). This is also true for m-convex Q-algebras [9]. In this section we would like to prove this for an FLM algebras but for some special elements. By $G(\mathcal{A})$ we mean the set of all invertible elements of $\mathcal{A}$ and $S(A) = \{x \in \mathcal{A} : \exists\ n, b^n x^n \to 0\}$.

Theorem 4.1. *In an unital complete metrizable FLM algebra $\mathcal{A}$ if $\Phi_{\mathcal{A}}$ separates the points of $\mathcal{A}$ for every element $x \in S(\mathcal{A})$, then $sp(x)$ is a compact and nonvoid subset of $\mathbb{C}$.*

Proof : Compactness is proved in [2]. Let $sp(x) = \phi$, then for every λ in $\mathbb{C}$, $(x - \lambda e)$ is invertible. So for an arbitrary $x \in \mathcal{A}$ define $R_x : \mathbb{C} \longrightarrow A$, with $R_x(\lambda) = (x - \lambda e)^{-1}$ for all $\lambda \in \mathbb{C}$.

Let φ be a multiplicative linear functional on $\mathcal{A}$ (i.e. an element of carrier space of $\mathcal{A}$) φ is continuous (theorem 4.5, [2]). We have

$$\varphi \circ (R_x(\lambda)) = \frac{1}{\varphi(x - \lambda e)},$$

for all $\lambda \in \mathbb{C}$. By (theorem 4.1, [2]) if for some $b > 1, b^n x^n \longrightarrow 0$ in $\mathcal{A}$ then

$$(1 - x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n,$$

suppose that $\lambda \geq 1$, so there exists some $b > 1$ such that

$$(\frac{b^n}{\lambda^n})x^n \longrightarrow 0 \qquad or \qquad b^n(\frac{x}{\lambda})^n \longrightarrow 0$$

and in this case we have

$$(\frac{x}{\lambda} - e)^{-1} = 1 + \sum_{n=1}^{\infty} x^n \lambda^{-n}$$

and so

$$(x - \lambda e)^{-1} = \lambda^{-1} + \sum_{n=1}^{\infty} x^n \lambda^{-n-1} = \sum_{n=0}^{\infty} x^n \lambda^{-n-1}.$$

Now because of continuity of φ we can write

$$\begin{aligned}\left|\varphi(x-\lambda e)^{-1}\right| &= \left|\varphi\left(\sum_{n=0}^{\infty}\frac{x^n}{\lambda^{n+1}}\right)\right| \leq \left|\sum_{n=0}^{\infty}\frac{\varphi(x^n)}{\lambda^{n+1}}\right| \\ &\leq \sum_{n=0}^{\infty}\frac{|\varphi(x^n)|}{|\lambda^{n+1}|} \leq \sum_{n=0}^{\infty}\frac{1}{|\lambda^{n+1}|} = \frac{1}{1-|\lambda|}\end{aligned}$$

Now for any x we have $|\varphi(x-\lambda e)^{-1}| \longrightarrow 0$, as $\lambda \longrightarrow \infty$. Then follows from Liovilles theorem that

$$\varphi \circ R_x(\lambda) = 0,$$

for all $\lambda \in \mathbb{C}$. Since Φ_A separates points of $\mathcal{A}$ we may conclude that $R_x(\lambda) = 0$ for all $\lambda \in \mathbb{C}$ but by our assumption $R_x(\lambda) \in G(\mathcal{A})$ and φ can not map any invertible element to 0. This contradict compleat the proof.□

Let $\mathcal{A}$ be a commutative complete metrizable FLM algebra. $\Phi_{\mathcal{A}}$ is denote the maximal ideal space of $\mathcal{A}$. For complete metrizable FLM algebra $\mathcal{A}$, the carrier space $\Phi_{\mathcal{A}}^{\infty}$ (maximal ideal space) is a locally Hausdorff space with one point compactification $\Phi_{\mathcal{A}}^{\infty}$. Also when $\mathcal{A}$ is unital, then $\Phi_{\mathcal{A}}$ is compact (theorem 5.6, [4]).

Theorem 4.2. *Let $\mathcal{A}$ be a commutative complete metrizable FLM algebra and $\mathcal{B}$ be a dense ideal of $\mathcal{A}$. Suppose that $\mathcal{B}$ is a complete metrizable FLM algebra with another topology (meter). Then $\Phi_{\mathcal{B}}$ is homeomorphic to $\Phi_{\mathcal{A}}$.*

Proof : Let $\widetilde{\varphi} \in \Phi_{\mathcal{A}}$. Since $\mathcal{B}$ is dense in $\mathcal{A}$, thus restriction of $\widetilde{\varphi}$ on $\mathcal{B}$ is in $\Phi_{\mathcal{B}}$. Therefore it is sufficient to show that every $\varphi \in \Phi_{\mathcal{B}}$ can be uniquely extend to $\widetilde{\varphi} \in \Phi_{\mathcal{A}}$.

Let $\varphi \in \Phi_{\mathcal{B}}$, then we have

$$\varphi(ab) = \widetilde{\varphi}(a)\widetilde{\varphi}(b) \qquad\qquad (a \in \mathcal{A}, b \in \mathcal{B}),$$

where $\widetilde{\varphi}$ is the extension of φ. For all $a \in \mathcal{A}$, define

$$\widetilde{\varphi}(a) := \frac{\varphi(ab)}{\varphi(b)},$$

for some $b \in \mathcal{B}$, with $\varphi(b) \neq 0$. It is clear that $\widetilde{\varphi}(a)$ does not depend on the choice of b. Let $a \in \mathcal{B}$, then

$$\widetilde{\varphi}(a) = \frac{\varphi(ab)}{\varphi(b)} = \frac{\varphi(a)\varphi(b)}{\varphi(b)} = \varphi(a),$$

this shows that $\widetilde{\varphi}$ is an extension of φ, where this extension is unique. Therefore $\widetilde{\varphi} \in \Phi_{\mathcal{A}}$.

The restriction map $\widetilde{\varphi} \longrightarrow \varphi$ is continuous, therefore $\Phi_{\mathcal{B}}$ is homeomorphic to $\Phi_{\mathcal{A}}$.$\square$

Now let $\mathcal{B}$ is not dense in $\mathcal{A}$, the homomorphism φ in maximal ideal of $\mathcal{B}$ is still uniquely extendable to a homomorphism $\widetilde{\varphi}$ in maximal ideal of $\mathcal{A}$ in the same way as above. Then we have the following theorem.

Theorem 4.3. *Let $\mathcal{A}$ be a commutative complete metrizable FLM algebra and $\mathcal{B}$ be an ideal of $\mathcal{A}$. Suppose that $\mathcal{B}$ is a complete metrizable FLM algebra with some topology. Then $\Phi_{\mathcal{B}}$ is homeomorphic to open subset of $\Phi_{\mathcal{A}}$.*

Proof : As in the proof of theorem 4.2, every nonzero homomorphism on $\mathcal{B}$ can be extended to homomorphism on $\mathcal{A}$ and this extension is unique.

Let $T : \Phi_{\mathcal{B}} \longrightarrow \Phi_{\mathcal{A}}$, with $T(\varphi) = \widetilde{\varphi}$. Since the extension of φ to $\widetilde{\varphi}$ is unique, therefore T is one to one and for the same reason in the proof of theorem 4.2, T is continuous.

T^{-1} with domain $T(\Phi_{\mathcal{B}})$ is continuous, thus T is a homeomorphism. Let $\widetilde{\varphi} \notin \Phi_{\mathcal{B}}$, then $\widetilde{\varphi}|_{\mathcal{B}} = 0$. It is clear that with $w^{*}-$topology $\bigcap\{\varphi \in \Phi_{\mathcal{A}} : \varphi|_{\mathcal{B}} = 0\}$ is closed in $\Phi_{\mathcal{A}}$. Hence $T(\Phi_{\mathcal{B}})$ is open.$\square$

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