International Journal of Algebra, Vol. 5, 2011, no. 3, 135 - 138

c-Maximal Ideal of Finite Rings

Mohammad Tashtoush and Musa Jawarneh

Department of Basic Sciences
AL-Balqa' Applied University, AL-Huson College University
P. O. Box 50, AL-Huson 21510

mhmdmath@hotmail.com musa_jawarneh@yahoo.com

Abstract

An ideal H in a finite ring R is called $c-$maximal ideal of a ring R if there exists an ideal N of R such that $R = HN$ and $H \cap N \leq H_R$, where $H_R = Core_R(H)$ is the maximal ideal of R which is contained in H.

In this paper we introduce the basic definition of c-maximal ideal of finite rings, and to studying some properties of c-maximal ideal of finite rings.

Keywords: maximal ideal, $c-$ maximal ideal, finite rings.

1. Introduction and Preliminaries

It is interesting to use some information on the subgroups of a group G to determine the structure of the group G. The normality of subgroups in a finite group plays an important role in the study of a finite group. Similarly, ideals and maximal ideals of rings play an important role to studying rings.

In this paper we will introduce the basic definition of $c-$maximal ideal of finite rings, and using the $c-$maximally property to investigate some conditions or properties of finite rings.

2. Elementary Properties

Definition 2.1: An ideal H is called $c-$maximal ideal of a ring R if there exists an ideal N of R such that $R = HN$ and $H \cap N \leq H_R$, where $H_R = Core_R(H)$ is the maximal ideal of R which is contained in H.

Example 2.2: Let $R = Z_p$ be the ring of integers modulo p (p prime) under addition and multiplication modulo p. Then $H = \langle p \rangle$ is $c-$maximal ideal of a ring R since there exists an ideal $N = \langle e \rangle$ of R such that $R = HN$ and $H \cap N \leq H_R$.

With return to reference [8], we can generalize the basic definition of c_s-normal subgroups of finite groups, for maximal ideals of finite rings. In this study, we need to prove the following theorems and properties for rings.

Dedekind's Identity for Rings 2.3: Let R be a ring with subrings A, B and C such that $B \leq A$ (B is a subring of A). Then $A \cap BC = B(A \cap C)$.

Proof: Certainly $B(A \cap C) \subseteq A \cap BC$ since $B \leq A$. Let $a \in A \cap BC$, then $a = bc$ for some $b \in B$ and $c \in C$. Then since $B \leq A$. Hence $a \in B(A \cap C)$. Thus $A \cap BC = B(A \cap C)$.

Theorem 2.4 [3]: Let I be a nonempty subset of a ring R. Then I is an ideal of R iff *(i)* If $a, b \in I$ then $a + b \in I$, *(ii)* If $a \in I$ then $-a \in I$, *(iii)* If $a \in I$ and $r \in R$ then ar and $ra \in I$.

Theorem 2.5: Let R be a ring with subrings A, B and C. Then:
(a) If B is an ideal of R, then $A \cap B$ is an ideal of A.
(b) If A and B are two ideals of R, then AB is an ideal of R.

Proof: (a) Let $x, y \in A \cap B$, since $A \leq R$ and B is an ideal of R, then $x + y \in A \cap B$ and if $x \in A \cap B$ then $-x \in A \cap B$. If $x \in A \cap B$ and $a \in A$ then ax and $xa \in A \cap B$. Therefore, $A \cap B$ is an ideal of A.

(b) The set AB defined by $\{ab : a \in A, b \in B\}$.

Let $x, y \in AB \Rightarrow x = a_1 b_1$ and $y = a_2 b_2$ for any elements $a_1, a_2 \in A$ and $b_1, b_2 \in B \Rightarrow x + y = a_1 b_1 + a_2 b_2 \in AB$. If $x \in AB$ then $-x = -(a_1 b_1) = -(-a_1)(-b_1) \in AB$. If $r \in R$ and $x \in AB$ then $rx = r(a_1 b_1) = (ra_1) b_1 = (a_1 r) b_1 = a_1 (rb_1) = a_1 (b_1 r) = (a_1 b_1) r = xr \in AB$.

Theorem 2.6: Let R be a ring with subring H and an ideal K such that $K \leq H \leq R$ and $K \leq N \leq R$. Then $R = HN$ iff $R/K = (H/K)(N/K)$.

Proof: Suppose that $R = HN$. Let $rK \in R/K$ for all $r \in R$ $\Rightarrow rK = (hn)K = (hK)(nK) \in (H/K)(N/K)$ for all $h \in H$ and $n \in N$. Therefore, $R/K \leq (H/K)(N/K)$. On the other hand, if $(hK)(nK) \in (H/K)(N/K)$ then $(hK)(nK) = (hn)K = rK \in R/K$. Hence $R/K = (H/K)(N/K)$.

Conversely, Suppose that $R/K = (H/K)(N/K)$, we want to show that

$R \le HN$. For all $rK \in \left(H/K\right)\left(N/K\right)$, there exists $h \in H$, $n \in N$ and $k \in K$ such that $r = hnk$ since $K \le H \cap N$ then $r = hnn' = hn_2$ where $n_2 = nn'$ so that for any $r \in R$ there exists $n_2 \in N$ and $h \in H$ such that $r = hn_2$ and hence $R \le HN$.

3. Theorem

Theorem 3.1: Let R be a ring. If H is an ideal in R, then H is $c-$maximal ideal in R .

Proof: Suppose that H is an ideal in R, since R is an ideal of itself, then $R = HR$ and $H \cap R = H \le H_R$. Hence H is $c-$maximal ideal in R .

Theorem 3.2: Let R be a ring with subrings H and K . If H is $c-$maximal ideal in R with $H \le K \le R$, then H is $c-$maximal ideal in K .

Proof: Suppose that H is $c-$maximal ideal in R . Then there exists an ideal N in R such that $R = HN$ and $H \cap N \le H_R$. By using theorem 2.3 we have that $K = K \cap G = K \cap HN = H\left(K \cap N\right)$. By theorem 2.5, part *a*, then $K \cap N$ is an ideal of K . Therefore,

$H \cap \left(N \cap K\right) = \left(H \cap N\right) \cap K \le H_R \cap K \le H_K$ since H_R is an ideal in R .

Theorem 3.3: Let K be an ideal in R with $K \le H$. Then H is $c-$maximal ideal in R iff H/K is $c-$maximal ideal in G/K .

Proof: Suppose that H/K is $c-$maximal ideal in R/K . Then there exists an ideal N/K in R/K such that $R/K = \left(H/K\right)\left(N/K\right)$ and $\left(H/K\right) \cap \left(N/K\right) \le \left(H/K\right)_{\left(R/K\right)}$. By theorem 2.6 we have that $R = HN$ and $H \cap N \le H_R$.

Conversely, suppose that H is $c-$maximal ideal in R . Then there exists an ideal N in R such that $R = HN$ and $H \cap N \le H_R$. By theorem 2.6 we have that $R/K = \left(H/K\right)\left(NK/K\right)$ and hence by using theorem 2.5, part *b*, then NK is an ideal in R , and

$$\left(H/K\right) \cap \left(NK/K\right) = \left(H \cap NK\right)/K = K\left(H \cap N\right)/K \le KH_R/K = \left(H/K\right)_{\left(R/K\right)}.$$

Therefore, H/K is $c-$maximal ideal in R/K .

References

[1] Allenby, R. B. J. T., **Rings, Fields and Groups; An Introduction to Abstract Algebra,** 2nd Edition, Routledge, Chapman and Hall, Inc., New York, 1991.

[2] Doerk, K. and Hawakes, T. O., **Finite Soluble Groups**, Walter de Gruyter, Berlin, 1992.

[3] Dummit, D. S. and Foote, R. M., **Abstract Algebra**, 2nd Edition, John Wiley and Sons, Inc., New York, 1999.

[4] Henry, B. G., **Between Nilpotent and Solvable**, Polygonal Publishing House, New Jersey, 1982.

[5] Lujin Zhu, Weubin Guo & K. P. Shum, **Weakly c- Normal Subgroups of Finite Groups and Their Properties**, Communications in Algebra, Vol. 24, PP 5505-5512 (2002), Article No. 11.

[6] Miao Long, Chen Xiaoli & Guo Wenbin, **On c- Normal Subgroups of Finite Groups**, Southeast Asian Bulletin of Mathematics, Vol. 25, PP 479-483 (2001).

[7] Rose, J. A., **A Course on Group Theory**, Cambridge University Press, London, 1978.

[8] Tashtoush, M., **Weakly c-Normal and cs-Normal Subgroups of Finite Group**s, Jordan Journal of Mathematics and Statistics (JJMS), Vol. 1, No (2), PP 123-132, Article No. 3, 2008.

[9] Wang, Yanming, **c- Normality of Groups and its Properties**, Journal of Algebra, Vol. 180, PP 954-965 (1996), Article No. 103.

Received: August, 2010