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t-Norm $(\lambda,\ \mu)$-Fuzzy Sub Near-Rings and t-Norm $(\lambda,\ \mu)$-Fuzzy Ideals of Near Rings

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Abstract

In this paper, we introduce the concept of t-norm $(\lambda,\ \mu)$-fuzzy sub near-ring which can be regarded as a generalisation of t-norm $(\lambda,\ \mu)$-fuzzy sub ring and t-norm $(\lambda,\ \mu)$-fuzzy ideal of a near-ring.

Key words and phrases: t-norm $(\lambda,\ \mu)$-fuzzy sub near-ring, t-norm $(\lambda,\ \mu)$-fuzzy ideal of a near-ring, homomorphism, isomorphism

1 Introduction

Throughout this paper N stands for near-ring. For basic terminology and notations for nearring we refer to Piln [7]. We expand the notion of t-norm $(\lambda,\ \mu)$-fuzzy ideals for rings to t-norm $(\lambda,\ \mu)$-fuzzy ideals of near-rings from Rajesh Kumar [8]. The concept of fuzzy sets was introduced by Zadeh [12] in 1965. Rosenfield [9] introduced the notion of a fuzzy group as easly as 1971. The notions of fuzzy subnear-ring and ideals were introduced by S-Abou-Zaid in 1991 [2, 3]. Basicaly this paper is devalaped from Bingxue Yao [4, 1].

Definition 1.1 A triangular norm, [8] t-norm is a function $t : [0,1] \times [0,1] \longrightarrow [0,1]$satisfying, for each $a, b, c, d, \in [0,1]$,the following conditions: (i) $t(0,0) = 0, t(a,1) = a$; (ii) $t(a,b) \leq t(c,d)$,whenever $a \leq c, b \leq d$; (iii) $t(a,b) = t(b,a)$;and (iv) $t(t(a,b),c) = t(a,t(b,c))$

Example 1.2 *A funcation $t : [0,1] \times [0,1] \longrightarrow [0,1]$,defined as $t(a,b) = ab$ is a t-norm.*

Example 1.3 *A funcation $t : [0,1] \times [0,1] \longrightarrow [0,1]$,defined as $t(a,b) = a \wedge b$ is a t-norm.*

Example 1.4 *A function $t : [0,1] \times [0,1] \longrightarrow [0,1]$, defined as*

$$t(a,b) = \begin{cases} a & \text{if b=1} \\ b & \text{if a=1} \\ 0 & \text{otherwise} \end{cases}$$

is a t-norm.

2 t-norm $(\lambda,\ \mu)$-fuzzy subnear-ring and t-norm $(\lambda,\ \mu)$-fuzzy ideal of a near-rings

Based on the concepts of fuzzy subgroup with thresholds introduced by X.Yuan [11] and $(\lambda,\ \mu)$-fuzzy normal subgroup introduced by B. Yao [10], we introduce the following concept. In the following discussion, we always assume that $0 \leq \lambda < \mu \leq 1$.

Definition 2.1 A fuzzy set A in N is $t : N \longrightarrow [0,1]$

Definition 2.2 A fuzzy set A of near-ring N is said to be $(\lambda,\ \mu)$-fuzzy sub near-ring, if (i)$f(x+y) \vee \lambda \geq t[t(f(x), f(y)), \mu]$, for all $x, y \in N$, (ii)$f(xy) \vee \lambda \geq t[t(f(x), f(y)), \mu]$, for all $x, y \in N$

Every $(\lambda,\ \mu)$ fuzzy ideal is always a t-norm$(\lambda,\ \mu)$-Fuzzy ideal, by taking t-norm as $t(a,b) = min(a,b)$

Example 2.3 *Consider the ring Z_6. Let $\lambda = .3$ and $\mu = .6$ and $A : Z_6 \longrightarrow [0,1]$ defined by $A(0) = .7, A(3) = .5, A(4) = .1, A(2) = .1, A(1) = .2, A(5) = .1$. This is $(\lambda,\ \mu)$-fuzzy ideal. Therefore $(\lambda,\ \mu)$-fuzzy ideal is a generalization of fuzzy ideals. The above example is a $(\lambda,\ \mu)$-fuzzy ideal but not a fuzzy ideal, since $A(2) = A(1+1) = .1,\ A(1) \wedge A(1) = .2$. Thus $A(1+1) \not\geq A(1) \wedge A(1)$. Therefore A is not a fuzzy ideal, but A is a $(\lambda,\ \mu)$-fuzzy ideal.*

Definition 2.4 Let A be a fuzzy subset of N. Then A is called a t-norm $(\lambda,\ \mu)$-fuzzy sub near-ring of N if for all $x, y \in N$ (i) $A(x+y) \vee \lambda \geq t(t(A(x), A(y)), \mu)$, (ii)$A(-x) \vee \lambda \geq\ t(A(x), \mu)$, (iii)$A(xy) \vee \lambda \geq t(t(A(x), A(y)), \mu)$,Where $0 \leq \lambda, \mu \leq 1$. ■

Proposition 2.5 *Let A be fuzzy subset of N. Then A is a t-norm (λ, μ)-fuzzy sub near-ring of N if and only if for all $x, y \in N$ (i) $A(x-y) \vee \lambda \geq t(t(A(x), A(y)), \mu)$, (ii)$A(xy) \vee \lambda \geq t(t(A(x), A(y)), \mu)$.*

Proof. The proof is straightfarward and omitted. ■

Proposition 2.6 *Let A be t-norm $(\lambda,\ \mu)$- fuzzy sub near-ring of N. Then $A(0) \vee \lambda \geq t(A(x), \mu), \forall x \in N$.In particular, if there exists $x_0 \in N$ such that $A(x_0) \geq \mu$, then $A(0) \geq \mu$.*

Proof. Let A be a t-norm($\lambda,\ \mu$)- fuzzy subnear-ring of N. Then, $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu)$ put $y=x$, then $A(x-x)\vee\lambda \geq t(t(A(x),A(x)),\mu) \Rightarrow A(0)\vee\lambda \geq t(A(x),\mu), \forall x \in N$. Suppose $A(x_0) \geq \mu$ for some $x_0 \in N$. Then $A(0)\vee\lambda \geq t(A(x_0),\mu) \geq t(\mu,\mu) = \mu \Rightarrow A(0)\vee\lambda \geq \mu$. Therefore $A(0) \geq \mu$. (since $A(0)$ is allways grater). ■

Proposition 2.7 *Let A be a fuzzy subset of N. Then A is a t-norm $(\lambda,\ \mu)$- fuzzy sub near-ring of N if and only if A_c is a sub near-ring of N for all $c \in (\lambda,\ \mu)$.*

Proof. Let A be a t-norm $(\lambda,\ \mu)$- fuzzy sub near-ring of N. Let $c \in (\lambda,\ \mu)$ and $x, y \in A_c$, then $A(x) \geq c$ and $A(y) \geq c$. Consider $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu) \geq t(t(c,c),\mu) = c$. That is $A(x-y)\vee\lambda \geq c \Rightarrow A(x-y) \geq c$. (since $c > \lambda$)$\Rightarrow$ $x-y \in A_c$. Consider $A(xy)\vee\lambda \geq t(t(A(x),A(y)),\mu) \geq t(t(c,c),\mu) = c$. That is $A(xy)\vee\lambda \geq c \Rightarrow A(xy) \geq c$. (since $c > \lambda$),$\Rightarrow$ $xy \in A_c$. Therefore A_c is a subnear-ring of N. Conversely, let A_c be a sub near-ring of N, $\forall c \in (\lambda,\ \mu]$. Suppose $A(x-y)\vee\lambda < t(t(A(x),A(y)),\mu) = c$, then $A(x-y) < c$ (since$c > \lambda$)$\Rightarrow$ $x-y \notin A_c$ for $x, y \in A_c$, which is a contradiction to that A_c is a subnear-ring of N. Hence $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu)$ for all $x, y \in N$. Also $A(xy)\vee\lambda \geq t(t(A(x),A(y)),\mu) \geq t(t(c,c),\mu) = c$. Suppose $A(xy)\vee\lambda < t(t(A(x),A(y)),\mu) = c$ that is $A(xy)\vee\lambda < c \Rightarrow A(xy) < c$ (since $x > c$) $\Rightarrow xy \notin A_c$ for all $x, y \in A_c$ which is a contradiction. So $A(xy)\vee\lambda \geq t(t(A(x),A(y)),\mu)$. Therefore A is a $(\lambda,\ \mu)$-fuzzy sub near-ring of N. ■

Definition 2.8 Let A be a fuzzy subset of N. Then A is called a t-norm $(\lambda,\ \mu)$-fuzzy ideal of N if for all $x, y \in N$ (i) $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu)$, (ii)$A(x+y-x)\vee\lambda \geq t(A(y),\mu)$, (iii)$A(xy)\vee\lambda \geq t(A(x),\mu)$, (iv)$A(x(y+i)-xy)\vee\lambda \geq t(A(i),\mu)$ ■

Proposition 2.9 *Let A be a fuzzy subset of N. Then A is a t-norm $(\lambda,\ \mu)$- fuzzy ideal of N if and only if A_c is a ideal of N for all $c \in (\lambda,\ \mu]$.*

Proof. Assume that A is a t-norm $(\lambda,\ \mu)$- fuzzy ideal of N. To prove A_c is a ideal of N. Let $x, y \in A_c$. Then $A(x) \geq c$ and $A(y) \geq c$. Consider $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu) \geq t(t(c,c),\mu) = c$. (since$c_i < \lambda$), that is $A(x-y)\vee\lambda \geq c \Rightarrow A(x-y) \geq c$ (since $c < \lambda$),$\Rightarrow$ $x-y \in A_c$. Let $y \in A_c$, To prove $x+y-x \in A_c$. Consider $A(x+y-x)\vee\lambda \geq t(A(y),\mu) \geq t(c,\mu) = c$. That is $A(x+y-x)\vee\lambda \geq c \Rightarrow A(x+y-x) \geq c$, (Since $c > \lambda$)$\Rightarrow$ $x+y-x \in A_c$. Let $y \in A_c$, To prove $x+y-x \in A_c$. Consider $A(x+y-x)\vee\lambda \geq t(A(y),\mu) \geq t(c,\mu) = c$. That is $A(x+y-x)\vee\lambda \geq c \Rightarrow A(x+y-x) \geq c$ (since $c > \lambda$)$\Rightarrow$ $x+y-x \in A_c$. Let $x \in A_c$ and $y \in N$. Then $A(xy)\vee\lambda \geq t(A(x),\mu) \geq t(c,\mu) \geq c.A(xy)\vee\lambda \geq c \Rightarrow A(xy) \geq c$. (since $c > \lambda$) $\Rightarrow xy \in A_c$. Let $i \in A_c$, then $A(i) \geq c$ and $x(y+i)-xy \in c.A(x(y+i)-xy)\vee\lambda \geq t(A(i),\mu) \geq t(c,\mu) = c. \Rightarrow A(x(y+i)-xy)\vee\lambda \geq c \Rightarrow A(x(y+i)-xy) \geq c$(since $\lambda < c$) $\Rightarrow (x(y+i)-xy) \in A_c$. Therefore A_c is a ideal of N. Conversely, assume that A_c is a ideal of N for all $c \in (\lambda,\ \mu]$. We have to prove four conditions as follows. (i) Let $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu)$. Suppose let us consider,$A(x-y)\vee\lambda < t(t(A(x),A(y)),\mu) = c$, then $A(x-y)\vee\lambda < c \Rightarrow A(x-y) < c \Rightarrow x-y \notin A_c$ for $x,y \in A_c$. Which is contradiction of A_c is ideal. Thearfore $A(x-y)\vee\lambda \geq t(t(A(x),A(y)),\mu)$. (ii) Suppose $A(x+y-x)\vee\lambda \leq t(A(y),\mu) = c$, then $A(x+y-x)\vee\lambda \leq c \Rightarrow A(x+y-x) \leq c \Rightarrow x+y-x \notin A_c$. But $y \in A_c$ and A_c is ideal. We get the contradiction. Therefore $A(x+y-x)\vee\lambda \geq t(A(y),\mu)$. (iii) Suppose $A(xy)\vee\lambda < t(t(A(x),A(y)),\mu) = c$, then $A(xy)\vee\lambda < c \Rightarrow A(xy) < c \Rightarrow xy \notin A_c$, which is a contrdiction. Therefore $A(xy)\vee\lambda \geq t(A(x),\mu)$. (iv) Suppose $A(x(y+i)-xy)\vee\lambda < t(A(i),\mu) = c$, then $A(i) \geq t \Rightarrow i \in A_c$. Now $A(x(y+i)-xy)\vee\lambda < c \Rightarrow A(x(y+i)-xy) < c$, (since $\lambda < c$) $\Rightarrow x(y+i)-xy \notin A_c$, which is a contradiction. Hence A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of N. Hence the theorem. ■

Proposition 2.10 *Let$f: N_1 \to N_2$ be a homomorphism of near-rings and let A be a t-norm $(\lambda,\ \mu)$- fuzzy sub near-ring ofN_1. Then f(A) is a t-norm $(\lambda,\ \mu)$- fuzzy sub near-ring of N_2.If A is a t-norm $(\lambda,\ \mu)$- fuzzy ideal of N_1 and f is onto, then f(A) is a t-norm $(\lambda,\ \mu)$- fuzzy ideal of N_2, where$f(A)(y) = sup\left\{A(x)/f(x) = y\right\},\ \forall y \in N_2$.*

Proof. (1) Let A be a t-norm $(\lambda,\ \mu)$- fuzzy subnear-ring ofN_1. To prove f(A) is a t-norm $(\lambda,\ \mu)$- fuzzy subnear-ring of N_2. Given that $f(A)(y) = sup\left\{A(x)/f(x) = y\right\}, \forall y \in N_2$. For this we have to show first, (i) for all $y_1, y_2 \in N_2$, we have $f(A)(y_1-y_2)\vee\lambda = sup\{A(x_1-x_2)/f(x_1-x_2) = y_1-y_2\}\vee\lambda = sup\{A(x_1-x_2)\vee\lambda f(x_1-x_2) = y_1-y_2\} \geq sup\{t\{A(x_1),A(x_2),\mu\}/f(x_1) = y_1, f(x_2) = y_2\}$ (since A is $(\lambda,\ \mu)$- fuzzy subnear-ring) $= sup\{A(x_1)/f(x_1) = y_1\}\vee\lambda = t\{sup\{A(x_2)/f(x_2) = y_2\},\mu\}$. That is $f(A)(y_1-y_2)\vee\lambda \geq t\{f(A)(y_1),$
$f(A)(y_2),\mu\}$. (ii)$f(A)(y_1y_2)\vee\lambda = sup\{A(x_1x_2)/f(x_1x_2) = y_1y_2\}\vee\lambda = sup\{A($
$x_1x_2)\vee\lambda/f(x_1)f(x_2) = y_1y_2\} \geq sup\{t\{A(x_1),A(x_2),\mu\}/f(x_1) = y_1, f(x_2) = y_2\} = sup\{A(x_1)/f(x_1) = y_1\}\vee\lambda = t\{sup\{A(x_2)/f(x_2) = y_2\},\mu\}$. That is $f(A)(y_1y_2)\vee\lambda \geq t\{f(A)(y_1),f(A)(y_2),\mu\}$. (2) Now assume that A is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of N_1. To prove f(A) is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of N_2. (i) By part one the proof is optain. (ii) To prove f(A) is t-norm $(\lambda,\ \mu)$-fuzzy normal $f(A)(y+i'-y)\vee\lambda=$ $\sup\{A(x+i-x)/f(x+i-x) = y+i'-y\} = sup\{A(x+i-x)\vee\lambda/f(x) = y, f(i) = i'\} \geq sup\{t\{A(i),\mu\}/f(i) = i'\} \geq sup\{t\{A(i),\mu\}/f(i) = i'\} = t\{sup\{A(i)/f(i) = i'\},\mu\} = \{f(A)(i'),\mu\}$ (iii) To prove f(A) is t-norm $(\lambda,\ \mu)$-fuzzy right ideal. Let $y, y' \in N_2$. Then, $f(A)(yy')\vee\lambda = sup\left\{A(xx')/f(xx') = yy'\right\}\vee\lambda = sup\left\{A(xx')\vee\lambda/f(xx') = yy'\right\} = sup\{A(xx')\vee\lambda/f(x) = y, f(x') = y'\} \geq t\{sup\{t\{A(x),\mu\}/f(x) = y\},\mu\} = sup\{A(x)/f(x) = y\}$.That is $f(A)(yy^1)\vee\lambda \geq t\{f(A)(y),\mu\}$ (iv) To prove f(A) is t-norm $(\lambda,\ \mu)$-fuzzy left ideal. Let $y, y' \in N_2$. Then, $f(A)(y_1(y_2+i')-y_1y_2)\vee\lambda = sup\{A(x_1(x_2+i)-x_1x_2)/f(x_1(x_2+i)-x_1x_2) = y_1(y_2+i')-y_1y_2\}\vee\lambda = sup\{A(x_1(x_2+i)-x_1x_2)\vee\lambda/f(x_1) = y_1, f(x_2) = y_2, f(i) = i'\} \geq sup\{t\{A(i),\mu\}/f(i) = i'\} = \{A(i)/f(i) = i'\},\mu\} = t\{f(A)(i'),\mu\}$. Therefore f(A) is a t-norm $(\lambda,\ \mu)$-fuzzy left ideal. Hence f(A) is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of N_2. ■

Proposition 2.11 *Let* $f : N_1 \to N_2$ *be a homomorphism of near-rings and let* B *be a* t*-norm* $(\lambda,\ \mu)$*-fuzzy subnear-ring (fuzzy ideal) of* N_2*. Then* $f^{-1}(B)$ *is a* t*-norm* $(\lambda,\ \mu)$*-fuzzy subnear-ring (fuzzy ideal) of* N_1*, where* $f^{-1}(B)(x) = B(f(x)) \forall x \in N_1$*.*

Proof. (1) To prove $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy subnear-ring. Given that $f^{-1}(B)(x) = B(f(x)) \forall x \in N_1$. First we have to prove (i) $f^{-1}(B)(x_1 - x_2) \vee \lambda \geq t\{t\{f^{-1}(B(x_1)), f^{-1}(B(x_2))\}, \mu\}$. For $x_1, x_2 \in N_1$. we have $f^{-1}(B)(x_1 - x_2) \vee \lambda = B(f(x_1 - x_2)) \vee \lambda = B(f(x_1) - (f(x_2)) \vee \lambda \geq t\{t\{B(f(x_1)), B(f(x_2))\}, \mu\} = t\{t\{f^{-1}(B(x_1)), f^{-1}(B(x_2))\}, \mu\}$. (ii) To prove $f^{-1}(B)(x_1x_2) \vee \lambda \geq t\{t\{f^{-1}(B(x_1)), f^{-1}(B(x_2))\}, \mu\}$ Consider $f^{-1}(B)(x_1x_2) \vee \lambda = B(f(x_1x_2)) \vee \lambda = B(f(x_1)f(x_2)) \vee \lambda \geq t\{t\{B(f(x_1)), B(f(x_2))\}, \mu\}$ (since B is a fuzzy ideal)$=t\{t\{f^{-1}(B(x_1)), f^{-1}(B(x_2))\}, \mu\}$. Hence $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy subnear-ring. (2)

To prove $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal. (i) $f^{-1}(B)(x_1 - x_2) \vee \lambda \geq t\{t\{f^{-1}(B(x_1)), f^{-1}(B(x_2))\}, \mu\}$, $\forall\ x, y \in N_1$. (ii)To prove $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy normal subgroup. We have $f^{-1}(B)(x + y - x) \vee \lambda = B(f(x + y - x)) \vee \lambda = B(f(x) + f(y) - f(x)) \vee \lambda \geq t\{B(f(y)), \mu\}$ (since B is (λ, μ)-fuzzy ideal) $= t\{f^{-1}(B)(y), \mu\}$. That is $f^{-1}(B)(x + y - x) \vee \lambda \geq t\{f^{-1}(B)(y), \mu\}$ Therefore $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy normal subgroup. (iii) To prove $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy right ideal. We have $f^{-1}(B)(xy) \vee \lambda = B(f(xy)) \vee \lambda = B(f(x)f(y)) \vee \lambda$ (since f is a normal)$\geq$ $t\{B(f(x)), \mu\}$ (since B is (λ, μ)-fuzzy ideal) $= t\{f^{-1}(B)(x), \mu\}$. (since B is (λ, μ)-fuzzy ideal) Therefore $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy right ideal. (iv) To prove $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy left ideal. we have $f^{-1}(B)(x(y + i) - xy) \vee \lambda = B(f(x(y+i)-xy)) \vee \lambda = B(f(x)(f(y)+f(i))-f(x)f(y)) \vee \lambda$. That is $f^{-1}(B)(x(y+i) - xy) \vee \lambda \geq t\{B(f(i)), \mu\}$ (since B is (λ, μ)-fuzzy ideal). Therefore $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy left ideal. Hence $f^{-1}(B)$ is a t-norm $(\lambda,\ \mu)$-fuzzy ideal of N. ■

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