International Journal of Algebra, Vol. 5, 2011, no. 3, 107 - 115

The Number of Homomorphic Images of an Abelian Group

Greg Oman

Ohio University, 321 Morton Hall
Athens, OH 45701, USA
ggoman@gmail.com

Abstract. We study abelian groups with certain conditions imposed on their homomorphic images. We begin by classifying the abelian groups which have but finitely many homomorphic images (up to isomorphism). We then determine the abelian groups G which have the maximum number of homomorphic images, in the sense that $G/H \ncong G/K$ whenever H and K are distinct subgroups of G.

Mathematics Subject Classification: 20K99

Keywords: finitely generated abelian group, finitely cogenerated abelian group, essential subgroup, quasi-cyclic group, divisible abelian group, p-height

1. Introduction

It follows easily from The Fundamental Theorem of Finitely Generated Abelian Groups that every finitely generated abelian group has but finitely many subgroups up to isomorphism. We enquire about the status of the converse:

Question 1. *If G is an abelian group with but finitely many subgroups up to isomorphism, then must G be finitely generated?*

It is not hard to find examples showing that the answer to this question is no. We present such an example below (which is surely well-known, but we include a proof).

Example 1. *Let G be the group of dyadic rational numbers under addition. Then G has exactly three subgroups up to isomorphism, yet G is not finitely generated.*

Proof: By definition, $G = \{\frac{a}{2^n} : a \in \mathbb{Z}, n \in \mathbb{Z}^+\}$. Since G is dense in $\mathbb{R}$, G is not finitely generated. Let $H \subseteq G$ be nontrivial. We will show that either $H \cong \mathbb{Z}$ or $H \cong G$. Consider $H \cap \mathbb{Z}$. It is easy to see that $H \cap \mathbb{Z}$ is a nontrivial subgroup of $\mathbb{Z}$, hence cyclic. Let $H \cap \mathbb{Z} := (h)$, and let $H' := \frac{1}{h}H$. We claim that

$$H' \subseteq G. \tag{1.1}$$

To see this, it suffices to show that $H \subseteq hG$. Thus let $\frac{a}{2^n} \in H$ be arbitrary. Then note that $a \in H \cap \mathbb{Z}$, whence $a \in (h)$. It follows that $a = mh$ for some $m \in \mathbb{Z}$. Hence $\frac{a}{2^n} = \frac{mh}{2^n} \in hG$. Now notice that since $h \in H$,

$$1 \in H'. \tag{1.2}$$

Lastly, we note trivially that

$$H \cong H'. \tag{1.3}$$

To finish the proof, it suffices to show that either H' is cyclic, or $H' = G$. For every positive integer n, we let $G_n := \{\frac{a}{2^n} : a \in \mathbb{Z}\}$. Note that $G_n \cong \mathbb{Z}$ for each n. If $H' \subseteq G_n$ for some positive integer n, then H' is cyclic and we are done. Thus we assume that $H' \nsubseteq G_n$ for every positive integer n and we show that $H' = G$. Let $n > 0$ be arbitrary. Since $H' \nsubseteq G_n$, there is some $x \in H' - G_n$. By (1.1), $x \in G$. Thus $x = \frac{a}{2^m}$ for some $a \in \mathbb{Z}$ and $m > 0$. Since $x \notin G_n$ and $\mathbb{Z} \subseteq G_n$, we may assume without loss of generality that $(2, a) = 1$. Also, since $x \notin G_n$, it is clear that $m > n$. Recall from (1.2) that $1 \in H'$, whence $\{\frac{a}{2^m}, \frac{2^m}{2^m}\} \subseteq H'$. Since $(2, a) = 1$, we conclude that $\frac{1}{2^m} \in H'$. We have shown that $\frac{1}{2^m} \in H'$ for arbitrarily large m, and hence $H' = G$. □

Having shown Question 1 to have a negative answer, it is natural to dualize this question by considering homomorphic images instead of subgroups. To give the reader the appropriate context, we quickly review a couple of definitions. Recall that if G is an abelian group and H is a subgroup of G, then H is said to be *essential for* G provided every nontrivial subgroup of G has nontrivial intersection with H. G is said to be *finitely cogenerated* provided G possesses a finite essential subgroup. Kurosh and Yahya provided a complete characterization of these groups. Before stating their result, we remark on notation. If n is a positive integer, then $\mathcal{C}(n)$ will denote the cyclic group with n elements. If p is a prime, then the *quasi-cyclic group of type* p, denoted $\mathcal{C}(p^\infty)$, is the direct limit of of the cyclic groups $\mathcal{C}(p^n)$ as $n \to \infty$.

Proposition 1 (Kurosh and Yahya)**.** *Let G be an abelian group. Then the following are equivalent:*
(i) G is finitely cogenerated.
(ii) G is a finite direct sum of quasi-cyclic groups and finite cyclic groups.
(iii) Every decreasing chain of subgroups of G stabilizes.

Proof: See Theorem 25.1, p. 110 of Fuchs [1]. □

As Example 1 shows, the property of having but finitely many subgroups up to isomorphism does not characterize the finitely generated abelian groups. However, the dual property of having but finitely many homomorphic images up to isomorphism actually *does* characterize the finitely cogenerated abelian groups. For the purposes of this paper, we shall call such abelian groups *quotient-finite*. The principal result of the next section is the following theorem:

Theorem 1. *Let G be an abelian group. Then G is quotient-finite if and only if G is finitely cogenerated.*

Thus we have a characterization of the abelian groups which have 'few' homomorphic images. In the last section, we answer the question of which groups have 'many' homomorphic images. The details follow.

Let G be an abelian group. It was shown some time ago in Szele [4] that distinct subgroups of G are not isomorphic if and only if G is a subgroup of the direct sum of the quasi-cyclic groups $\mathcal{C}(p^\infty)$, one for each prime p. More concretely, distinct subgroups of G are not isomorphic if and only if G is a subgroup of the (multiplicative) group of all complex roots of unity. More recently, William Weakley extends Szele's result to modules over a commutative Noetherian ring in Weakley [5], calling such modules *terse*. In the final section, we consider the dual question for abelian groups. Keeping with Weakley's terminology, we call an abelian group G *quotient-terse* provided whenever H and K are distinct subgroups of G, then $G/H \not\cong G/K$. We prove the following theorem:

Theorem 2. *Let G be an abelian group. Then G is quotient-terse if and only if one of the following holds:*

(i) $G \cong \bigoplus_{i>0} \mathcal{C}(p_i^{n_i})$, where each n_i satisfies $0 \leq n_i < \infty$ and (p_i) is an enumeration of the primes.

(ii) G is torsion-free of rank one and without elements of infinite height.

2. A Classification of the Quotient-Finite Abelian Groups

We begin by recalling some definitions and results from the literature which will be needed in our proof. An abelian group G is *p-divisible* (p a prime) provided $pG = G$. G is said to be *divisible* provided $nG = G$ for every positive integer n. It is easy to see that G is divisible if and only if G is p-divisible for every prime number p. We state the following well-known facts about divisible abelian groups.

Fact 1 (Structure Theorem for Divisible Abelian Groups)**.** *Every divisible abelian group is a direct sum of copies of $\mathbb{Q}$ and $\mathcal{C}(p^\infty)$ for various primes p. Conversely, every such direct sum is divisible.*

Proof: See Theorem 23.1, page 104 of Fuchs [2]. □

Fact 2. *Let D be a divisible abelian group. If D is a subgroup of the abelian group H, then D is a direct summand of H.*

Proof: See Theorem 21.2, page 100 of [2]. □

The next fact (due to Prüfer and Baer) is a structure theorem for the so-called bounded abelian groups.

Fact 3. *Let G be an abelian group. Suppose further that m is a positive integer with $mG = \{0\}$. Then G is a direct sum of finite cyclic groups.*

Proof: See Theorem 11.2, page 44 of [1]. □

We are almost ready to prove Theorem 1. We first establish a lemma.

Lemma 1. *The following hold:*
(i) The quotient-finite abelian groups are closed under homomorphic images.
(ii) $\mathbb{Q}$ is not quotient-finite.

Proof: (i) Trivial.
(ii) Let p be an arbitrary prime. Note that:

$$\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathcal{C}(p^\infty)$$

As each $\mathcal{C}(p^\infty)$ is a homomorphic image of $\mathbb{Q}$, we see that $\mathbb{Q}$ is not quotient-finite. □

Proof of Theorem 1: Suppose first that G is finitely cogenerated. We will show that G has but finitely many homomorphic images up to isomorphism. By Proposition 1, G is a finite direct sum of quasi-cyclic groups and finite cyclic groups. Since G is a direct sum of its p-components (i.e. the subgroups of G consisting of the elements whose orders are powers of p, where p is prime), we may assume without loss of generality that G is a p-group. Write $G := D \bigoplus H$, where D is divisible and H is a finite sum of cyclic groups whose orders are a power of p. Thus D is simply the direct sum of all $\mathcal{C}(p^\infty)$ summands of G, if there are any. Let us suppose that G has rank n and let $\varphi : G \to K$ be an epimorphism. Then rank(K)$\leq n$. Further, $\varphi(D)$ is a divisible p-subgroup of K, whence is a direct sum of at most n copies of $\mathcal{C}(p^\infty)$ by The Structure Theorem for Divisible Abelian Groups. By Fact 2, we have $K = \varphi(D) \bigoplus H'$ for some subgroup H' of K. Since H is a finite sum of cyclic p-groups, there is a positive integer m such that $mH = \{0\}$. We claim that also $mH' = \{0\}$. To see this, let $x \in H'$ be arbitrary. Then $x = \varphi(d+h)$ for some $d \in D$ and $h \in H$. Hence $mx = m\varphi(d + h) = \varphi(md + mh) = \varphi(md)$, and thus $mx \in \varphi(D) \cap H'$. By the directness of the sum, we conclude $mx = 0$. Thus H' is a direct sum of at most n cyclic p-groups (each of order less than or equal to m) by Fact 3. It is now clear that there are but finitely many possibilities for the structure of K up to isomorphism.

Conversely, suppose that G has but finitely many homomorphic images up to isomorphism. We will show that G is a finite direct sum of quasi-cyclic groups and finite cyclic groups, hence finitely cogenerated by Proposition 1. We first claim that $pG \neq G$ holds for only finitely many primes p. For suppose that p is prime and $pG \neq G$. Then G/pG is a nonzero abelian group all of whose nonzero elements have order p. It follows that G/pG is a vector space over $\mathbb{Z}/(p)$ and we have the following sequence of surjections:

$$G \to G/pG \cong \bigoplus_{i \in I} \mathbb{Z}/(p) \to \mathbb{Z}/(p)$$

Thus if there are infinitely many primes p for which $pG \neq G$, then there are infinitely many primes p for which $\mathbb{Z}/(p)$ is a homomorphic image of G, and this is a contradiction to our assumption. Let $S := \{p_1, p_2, \ldots, p_k\}$ be the set of primes p for which $pG \neq G$ (if $S = \varnothing$, then G is divisible; Lemma 1 and The Structure Theorem for Divisible Abelain Groups imply that G is a finite direct sum of quasi-cyclic groups). Let $p \in S$. We claim that there is a positive integer n for which $p^{n+1}G = p^n G$. For suppose this is not the case. Let $n > 0$ be arbitrary. Then $p^{n+1}G \subsetneq p^n G$. Let $x \in p^n G - p^{n+1}G$. Then $x = p^n g$ for some $g \in G$. Consider the image $\overline{g}$ of g in $G/p^{n+1}G$. Note that $p^{n+1}\overline{g} = \overline{0}$ but $p^n \overline{g} \neq \overline{0}$. It follows that $\overline{g}$ has order p^{n+1}. Note further that if $x \in G/p^{n+1}G$ has order p^i, then $i \leq n+1$. Since n was arbitrary, it follows that for each positive integer n, G has a homomorphic image K with an element $h \in K$ of order p^{n+1} but no elements of order p^m for $m > n+1$. We conclude that G has infinitely many homomorphic images up to isomorphism, a contradiction. Thus for each $p_i \in S$, there is a positive integer s_i with $p_i^{s_i+1}G = p_i^{s_i}G$. In particular, $p_i^{s_i}G$ is p_i-divisible. It follows easily that $p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}G$ is a divisible abelian group. Let $H := p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}G$ and let $n := p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$. Note trivially that $n(G/H) = \{0\}$, and thus it follows from Fact 3 that G/H is a direct sum of finite cyclic groups. Since H is a divisible abelian group and $H \subseteq G$, H is a direct summand of G (Fact 2). Thus $G = H \bigoplus K$ for some abelian group K. But note that this implies that $K \cong G/H$. Thus K is a direct sum of finite cyclic groups. Since H is divisible, it follows from The Structure Theorem for Divisible Abelian Groups that H is a direct sum of quasicyclic groups Further, $\mathbb{Q}$ cannot be included in the sum by Lemma 1. Thus G is a direct sum of quasicyclic groups and finite cyclic groups. Since G is quotient-finite, it is clear that there are only finitely many summands. Proposition 1 implies that G is finitely cogenerated, and the proof is complete. □

Now that we have this theorem in hand, we can easily generalize to cancellative commutative semigroups. Let S be a commutative semigroup. Recall that S is *cancellative* provided $a + b = a + c$ implies $b = c$ for all $a, b, c \in S$. It is well-known that each commutative cancellative semigroup S embeds into its group of quotients $G(S)$ (see for example Grillet [3], Proposition 3.2). The definition of quotient-finite extends naturally to semigroups, and the following relationship exists between S and $G(S)$.

Lemma 2. *Suppose that S is a cancellative commutative quotient-finite semigroup. Then the group of quotients $G(S)$ is also quotient-finite.*

Proof: We suppose that S is a cancellative commutative quotient-finite semigroup. Let $f : G(S) \to H$ and $g : G(S) \to K$ be group epimorphisms and suppose that $H \ncong K$. As S embeds canonically into $G(S)$, we may view S as a subsemigroup of $G(S)$. Hence the maps f and g give rise to semigroup homomorphisms of S into H and K, respectively. Let $H(S)$ and $K(S)$ denote the images of S in H and in K, respectively. We will show that $H(S) \ncong K(S)$. Clearly the group of quotients of $H(S)$ is H and the group of quotients of $K(S)$ is K. If $H(S)$ and $K(S)$ were isomorphic semigroups, then their quotient groups H and K would also be isomorphic, and this contradicts our assumption. Thus $H(S) \ncong K(S)$. Since S is quotient-finite, the preceding argument shows that $G(S)$ must also be quotient-finite. $\square$

Thus if S is a cancellative commutative quotient-finite semigroup, then Theorem 1 implies that $G(S)$ is a finite direct sum of finite cyclic and quasicyclic groups. In particular, $G(S)$ is torsion. It follows that S is a group, and thus $S = G(S)$. Using Proposition 1, the following corollary is immediate.

Corollary 1. *Let S be a cancellative commutative semigroup. The following are equivalent:*
(i) S is quotient-finite.
(ii) S is a group whose subgroups satisfy the minimum condition.
(iii) S is a finite direct sum of quasi-cyclic groups and finite cyclic groups.

3. A Classification of the Quotient-Terse Abelian Groups

We now change gears and recall that an abelian group G is *quotient-terse* provided $G/H \ncong G/K$ for distinct subgroups H and K of G. Analogous to the last section, we will classify all quotient-terse abelian groups. We begin with two lemmas.

Lemma 3. *Suppose that G is a quotient-terse abelian group. Then so is every homomorphic image of G.*

Proof: Suppose that G is quotient-terse, and consider a quotient group G/H. We suppose that K/H and K'/H are two distinct subgroups of G/H and we show that $(G/H)/(K/H) \ncong (G/H)/(K'/H)$. To see this, note that $(G/H)/(K/H) \cong G/K$ and $(G/H)/(K'/H) \cong G/K'$. Since $K \neq K'$ and G is quotient-terse, it follows by definition that $G/K \ncong G/K'$. $\square$

Lemma 4. *Let p be a prime, and let m and n be positive integers. Then the following groups are not quotient-terse:*
(i) The quasi-cyclic group $\mathcal{C}(p^\infty)$
(ii) $G := \mathcal{C}(p^n) \oplus \mathcal{C}(p^m)$

Proof: (i) It is well-known (see exercise 7, p. 67 of [1] for example) that the only nonzero homomorphic image of $\mathcal{C}(p^\infty)$ is $\mathcal{C}(p^\infty)$ itself, but $\mathcal{C}(p^\infty)$ contains the groups $\mathcal{C}(p^n)$ for every positive integer n. $\square$

(ii) Let $H := \mathcal{C}(p^n) \oplus \mathcal{C}(p^{m-1})$ and $K := \mathcal{C}(p^{n-1}) \oplus \mathcal{C}(p^m)$. Clearly H and K are distinct, yet $G/H \cong G/K \cong \mathcal{C}(p)$. $\square$

We use these lemmas to prove a proposition which will be a cornerstone of the proof of Theorem 2.

Proposition 2. *Suppose that G is quotient-terse and p is a prime. Then the p-component $G_p := \{g \in G\colon$ the order of g is a power of $p\}$ of G is cyclic.*

Proof: Before beginning the proof, we recall the following result of Kulikov: If G contains elements of order p, then either $\mathcal{C}(p^\infty)$ is a direct summand of G or $\mathcal{C}(p^k)$ is a direct summand of G for some positive integer k (this result appears as Corollary 24.3, p. 80 of [1]). Clearly we assume that G has elements of order p. If $\mathcal{C}(p^\infty)$ is a direct summand of G, then it is a homomorphic image of G. However, by Lemma 3, this implies that $\mathcal{C}(p^\infty)$ is quotient-terse, and we obtain a contradiction to Lemma 4. Hence $\mathcal{C}(p^k)$ is a direct summand of G for some positive integer k, whence $G = \mathcal{C}(p^k) \oplus H$ for some subgroup H of G. To finish the proof, it suffices to show that H does not possess elements of order p. If so, then by repeating the same argument, we obtain a direct summand of G of the form $\mathcal{C}(p^k) \oplus \mathcal{C}(p^l)$. Lemmas 3 and 4 now yield a contradiction, and the proof is complete. $\square$

Using the previous proposition, we show that all nontrivial torsion-free quotient-terse abelian groups are of rank one.

Lemma 5. *Suppose that G is a torsion-free quotient-terse abelian group. Then G has rank at most one.*

Proof: Let G be as stated and suppose by way of contradiction that G has rank greater than one. Let $x, y \in G$ be linearly independent over $\mathbb{Z}$, and let $(2x, 2y)$ be the subgroup of G generated by $2x$ and $2y$. Now consider the quotient-terse group $H := G/(2x, 2y)$. Note that $x \notin (2x, 2y)$: If so, then $x = 2nx + 2my$ for some integers n and m. This implies that $(2n-1)x + 2my = 0$. Since x and y are linearly independent over $\mathbb{Z}$ and G is torsion-free, we obtain $2n - 1 = 0$, which is impossible. Analogously, $y \notin (2x, 2y)$. But note that this implies that the image $\overline{x}$ of x in H has order 2. Similarly $\overline{y}$ is of order 2 in H. The linear independence of x and y implies (as above) that $\overline{x} \neq \overline{y}$. But now the 2-component of H cannot be cyclic (as it contains a subgroup isomorphic to $\mathcal{C}(2) \times \mathcal{C}(2)$), and this contradicts Proposition 2. $\square$

Using a similar argument, we prove that a quotient-terse abelian group cannot be mixed.

Lemma 6. *Suppose that G is a quotient-terse abelian group. Then G is either torsion or torsion-free.*

Proof: Suppose by way of contradiction that G is quotient-terse but G is neither torsion nor torsion-free. Let $x \in G$ have prime order p and let $g \in G$ be an element of infinite order. Consider the group $H := G/(pg)$. Since g has infinite order, $g \notin (pg)$. Thus $\overline{g}$ has order p in H. Since x is a nonzero torsion element, it is clear that $x \notin (pg)$. Thus $\overline{x}$ also has order p in H. We claim that $(\overline{g}) \cap (\overline{x}) = \{\overline{0}\}$. Suppose that m and n are integers and $m\overline{g} = n\overline{x}$. Then $mg - nx \in (pg)$. Thus there is some integer k such that $nx = (m - pk)g$. Since g has infinite order and x has finite order, the previous equation implies that $m - pk = 0$, and thus $nx = 0$. But then we obtain $n\overline{x} = \overline{0}$, and hence $m\overline{g} = n\overline{x} = \overline{0}$. We have now established $(\overline{g}) \cap (\overline{x}) = \{\overline{0}\}$. By Lemma 3, H is quotient-terse. By Proposition 2, the p-component of H is cyclic. However, H contains two distinct subgroups of order p, and so we have a contradiction. $\square$

We now revisit some results on torsion-free rank one abelian groups.

Definition 1. *Let G be an abelian group, $g \in G$ be nonzero, and let p be a prime. Consider the equation $p^k x = g$ as k runs through the nonnegative integers. If k is the greatest nonnegative integer for which it is solvable, then k is called the p-height of g. If no maximal k exists, g is said to have infinite p-height, and the height is then denoted by ∞. The height $H(g)$ of g is then defined to be the sequence of the p-heights of g as p runs through the primes. The element g is said to be of infinite height provided g has infinite p-height for some prime number p.*

We collect two important facts which will be used shortly (due to Beaumont and Zuckerman):

Lemma 7 ([1], remarks on p. 149)**.** *Suppose that G is a torsion-free abelian group of rank one. Then:*

(i) If $g \in G$ is nonzero and $H(g) = (k_1, k_2, \dots)$, then $G/(g) \cong \mathcal{C}(p_1^{k_1}) \oplus \mathcal{C}(p_2^{k_2}) \oplus \dots$.

(ii) Let H be a nonzero subgroup of G. Suppose that $h \in H$ has height $(k_1, k_2, \dots)$ in G and height $(l_1, l_2, \dots)$ in H. Then $l_i \leq k_i$ for each i, and

$$G/H \cong \mathcal{C}(p_1^{k_1 - l_1}) \oplus \mathcal{C}(p_2^{k_2 - l_2}) \oplus \dots$$

where $\infty - \infty = 0$ and $\infty - k = \infty$ for an integer k.

We are now ready to give a proof of Theorem 2. For notational ease, we enumerate the primes in increasing order as $p_1, p_2, p_3, \dots$.

Proof of Theorem 2: We consider first a group $G := \bigoplus_{i>0} \mathcal{C}(p_i^{n_i})$, where each n_i satisfies $0 \leq n_i < \infty$. Suppose that $H \neq K$ are subgroups of G. We will prove that $G/H \ncong G/K$. We have $H = \bigoplus_{i>0} \mathcal{C}(p_i^{k_i})$ and $K = \bigoplus_{i>0} \mathcal{C}(p_i^{l_i})$ where each k_i and l_i satisfy $0 \leq k_i, l_i \leq n_i$. Recall that each cyclic group $\mathcal{C}(p_i^{n_i})$ has a *unique* subgroup of order p_i^{α} for each $\alpha \leq n_i$. Since $H \neq K$, it follows that $k_i \neq l_i$ for some i. Note that the p_i component of G/H is (isomorphic to) $\mathcal{C}(p_i^{n_i - k_i})$ and the p_i component of G/K is $\mathcal{C}(p_i^{n_i - l_i})$. As $k_i \neq l_i$, it follows that these components are of different cardinalities, whence cannot be isomorphic. We conclude that $G/H \ncong G/K$.

We now consider a torsion-free rank one abelian group G without elements of infinite height. Let H and K be distinct subgroups of G. We will show that $G/H \ncong G/K$. Suppose first that $K = \{0\}$. Then $H \neq \{0\}$. Note that G/K is, of course, torsion-free. However, it follows from (ii) of Lemma 7 that G/H is torsion. Thus $G/H \ncong G/K$. Thus assume that H and K are both nonzero. Since G is rank one, $H \cap K \neq \{0\}$. It now follows from (ii) of Lemma 7 and the assumption that G is without elements of infinite height that $G/H \cap K$ is a direct sum of finite cyclic groups of prime power order with no repetition of primes in the summands. From what we have shown above, we see that $G/H \cap K$ is quotient-terse. Since $H \neq K$, we deduce that $H/H \cap K \neq K/H \cap K$, and hence $(G/H \cap K)/(H/H \cap K) \ncong (G/H \cap K)/(K/H \cap K)$. Note that $G/H \cong (G/H \cap K)/(H/H \cap K)$ and $G/K \cong (G/H \cap K)/(K/H \cap K)$. We conclude that $G/H \ncong G/K$.

Conversely, we suppose that G is an arbitrary quotient-terse abelian group and show that G falls into family (i) or (ii) from the statement of Theorem 2. By Lemma 6, G is either torsion or torsion-free. We suppose first that G is torsion. Then G is the direct sum of its p-components as p ranges over the primes. By Proposition 2, each p-component is cyclic, and we infer that G falls into family (i). We suppose now that G is torsion-free and nontrivial. By Lemma 5, it follows that G has rank one. We just need to show that G possesses no elements of infinite p height for any prime p. Suppose by way of contradiction that G possesses some element g of infinite p-height for some prime p. Then (i) of Lemma 7 shows that $\mathcal{C}(p^\infty)$ is a homomorphic image of G. By Lemma 3, it follows that $\mathcal{C}(p^\infty)$ is quotient-terse. This contradicts (i) of Lemma 4 and the proof is complete. □

We end the paper with the following question.

Question 2. *To what extent do Theorem 1 and Theorem 2 generalize to modules over a Noetherian domain?*

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Received: August, 2010