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Block Mutation of Block Quivers

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Abstract

In this paper, we introduce the concept of quivers with block structures and block mutation of their associated block quivers. The idea of block quivers arises from the concept of blocked exceptional collections in which similar exceptional objects are grouped together into blocks. Our study involves partitioning vertices of a quiver into blocks, where each block consists of similar vertices. Moreover, using the concept of quiver mutation introduced by Fomin and Zelevinsky, we study mutation of 3-block quivers in relation to mutation of blocked exceptional collections.

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1 Introduction

The study of blocks of quivers originates from the work on exceptional collections in which exceptional objects are grouped together into blocks. Karpov and Nogin [1] studied 3-block exceptional collections and showed that the three ranks corresponding to the collection satisfy a Markov-type Diophantine equation. They as well studied block mutation of exceptional collections and found that the mutant of a block mutation is similar to the exceptional collection obtained by mutating the collection at all the individual exceptional objects in the block. Also, they noted that this was independent of the order of mutations.

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Quivers can have similar vertices: no arrows joining them, same number of arrows coming in from a common vertex and same number of arrows leaving to a common vertex. In this paper, we study a partition called block structure on the vertices of the quiver in which similar vertices are grouped together into blocks. We then associate a block quiver to a quiver with block structure and define block mutation on it using the concept of quiver mutation introduced by Fomin and Zelevinsky [3] in the study of cluster algebras. Further, we show that block mutation of a 3-block quiver gives the same mutant as mutating the quiver at all the individual vertices in the block and this is independent of the order of mutations. We give basic concepts on quivers and their mutations in section 2 and our results are contained in sections 3 and 4.

2 Quivers and their mutations

A **quiver** is an oriented graph. Formally, a quiver is defined by $Q := (Q_0, Q_1, s, t)$ with Q_0 a set of vertices, Q_1 a set of arrows, $s : Q_1 \longrightarrow Q_0$ a map taking an arrow to its starting vertex, and $t : Q_1 \longrightarrow Q_0$ a map taking an arrow to its terminating vertex.

Moreover, if $s(a) = U$ and $t(a) = V$ for $U, V \in Q_0$, then a is an arrow from U to V and it is written as $a : U \longrightarrow V$. In this case, U and a as well as V and a are said to be **incident** to each other. We represent a quiver having m arrows from U to V by $U \xrightarrow{m} V$. Two quivers are considered to be **equivalent** if they are isomorphic as directed graphs. We denote two equivalent quivers, Q' and Q'', by $Q' \cong Q''$. We note here that, the quivers $U \xrightarrow{m} V$ and $W \xrightarrow{n} T$ are equivalent if they have the same number of arrows i.e $m = n$ for all $m, n \in \mathbb{N}$.

A quiver $Q' = (Q'_0, Q'_1, s', t')$ is called a **subquiver** of a quiver $Q = (Q_0, Q_1, s, t)$ if $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$ and where $t'(a) = t(a) \in Q'_0$, $s'(a) = s(a) \in Q'_0$ for any arrow $a \in Q'_1$. A subquiver Q' of a quiver Q is called **full** if for any two vertices U and V in Q', Q' contains all arrows between U and V present in Q. We shall denote a full subquiver of a quiver Q consisting of two vertices U and V by $Q_{U,V}$. For details on the definitions given above, see [2].

We call the quivers $V \circlearrowright$ and $U \rightleftarrows V$ a **loop** and a **2-cycle** respectively.

Mutation of quivers: In [3], Fomin and Zelevinsky defined the process of mutation of a quiver without loops and oriented 2-cycles incident to the vertex of mutation as follows: Given a quiver Q, mutation of Q at a vertex V involves the following steps:

(i) The vertices of the quiver remain unchanged.

(ii) All the arrows $c \in Q_1$ not incident to V remain unchanged.

(iii) Replace each arrow $a : U \longrightarrow V$ in Q by a new arrow $a^* : V \longrightarrow U$.

(iv) Replace each arrow $b : V \longrightarrow W$ in Q by a new arrow $b^* : W \longrightarrow V$.

(v) Add a new arrow $(ba) : U \longrightarrow W$ for each pair of arrows $a : U \longrightarrow V$ and $b : V \longrightarrow W$ in Q.

(vi) If there are arrows going both ways between two vertices, final number of arrows is their difference in the direction of the majority arrows.

The resultant quiver is said to be **reduced** and is called a **mutated quiver** (or a **mutant** of Q). We now proceed to give our results in the next sections.

3 Quivers with block structures

In this section we introduce the concept of quivers with block structures.

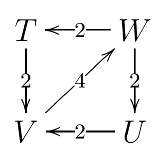
Definition 3.1. *We shall call a quiver* ***trivial*** *if it has no arrows.*

Definition 3.2. *Given a quiver $Q := (Q_0, Q_1, s, t)$, we call a subset B_ℓ of Q_0, a* ***block*** *if it satisfies the following properties:*

i) $\forall\, U, V \in B_\ell,\ \ Q_{U,V}$ *is trivial.*

ii) $\forall\, U, V \in B_\ell$ *and* $W \notin B_\ell,\ \ Q_{U,W} \cong Q_{V,W}$.

Example 3.3. Consider the quiver given below



Then $Q_0 = \{T, W, U, V\}$. We shall show that $B_\ell = \{T, U\}$ is a block for this quiver. $\forall\ U, T \in B_\ell,\ Q_{U,T}$ is trivial since there does not exist an arrow $a \in Q_1$ such that $s(a) = T$ and $t(a) = U$ or $s(a) = U$ and $t(a) = T$, so the first property holds. To show that the second property holds, we shall consider vertices $V, W \notin B_\ell$. Lets start with $V \notin B_\ell$. $Q_{V,U} = U \xrightarrow{2} V$ and $Q_{V,T} = T \xrightarrow{2} V$. Thus $Q_{V,U} \cong Q_{V,T}$. Now considering $W \notin B_\ell$, $Q_{W,U} = W \xrightarrow{2} U$ and $Q_{W,T} = W \xrightarrow{2} T$. Thus $Q_{W,U} \cong Q_{W,T}$. So the second property holds. Hence $B_\ell = \{T, U\}$ is a block.

Definition 3.4. *We define a* ***block structure*** *of a quiver Q to be a set $\mathcal{B} = \{B_j : j = 1, ..., n\}$ that satisfies the following properties:*

i) $\forall\ U \in Q_0\ \exists\ B_i$ *such that* $U \in B_i$.

ii) each B_j *is a block.*

iii) $B_i \cap B_k = \phi$ *where* $1 \leq i < k \leq n$.

Example 3.5. Consider the quiver in Example 3.3. Then $\mathcal{B} = \{\{T, U\}, \{V\}, \{W\}\}$ is a block structure for this quiver. Another block structure for this quiver is $\mathcal{B} = \{\{T\}, \{U\}, \{V\}, \{W\}\}$.

Lemma 3.6. *A block structure of a quiver* Q *is a partition of the vertices of the quiver into blocks, i.e.* $\forall\ U \in Q_0\ \exists!\ B_i$ *such that* $U \in B_i$.

Proof. Assume that $U \in B_1$ and $U \in B_2$. Then, $B_1 \cap B_2 \neq \phi$. This is a contradiction to Definition 3.4. Thus U cannot belong to both B_1 and B_2. This implies that $\exists!\ B_i$ such that $U \in B_i$. □

Definition 3.7. *A block structure in which each block consists of one vertex is called a* ***trivial block structure****. Otherwise the block structure is* ***non-trivial****.*

Example 3.8. For the quiver in Example 3.3, its trivial block structure is $\mathcal{B} = \{\{T\}, \{U\}, \{V\}, \{W\}\}$. Its nontrivial block structure is given as $\mathcal{B} = \{\{T, U\}, \{V\}, \{W\}\}$.

Remark 3.9. *A block structure of a quiver is not unique since the quiver can have both trivial and nontrivial block structures.*

Definition 3.10. *We shall call a quiver* Q *together with its block structure* $\mathcal{B}$*, a* ***quiver with block structure*** *and denote it by* $(Q, \mathcal{B})$.

Every quiver has a block structure so long as it has no loops as proved in the following lemma.

Lemma 3.11. *Quivers have no block structures if and only if they contain loops.*

Proof. Suppose that a quiver Q has no loops, then $\forall\ U \in Q_0$, the subquiver $Q_{U,U}$ is trivial. Also $\forall\ V \in Q_0 \setminus \{U\}$, $Q_{U,V} \cong Q_{U,V}$. Hence $\{U\}$ is a block. So choosing single vertices as blocks defines a block structure. Thus a quiver with no loops always has a trivial block structure. Conversely, suppose that Q contains a loop on $W \in Q_0$. Then $Q_{W,W}$ is not trivial. This implies that W cannot be in a block. Hence Q has no block structure. □

Definition 3.12. *A quiver with block structure* $(Q, \mathcal{B})$ *is an* ***n-block quiver*** *if its block structure consists of n blocks.*

Example 3.13. Consider the quiver in Example 3.3. If its block structure is chosen as $\mathcal{B} = \{\{T, U\}, \{V\}, \{W\}\}$ then the quiver is a 3-block quiver. But if its block structure is $\mathcal{B} = \{\{T\}, \{U\}, \{V\}, \{W\}\}$ then it is 4-block. Therefore the quiver is both 3-block and 4-block.

Lemma 3.14. *Trivial quivers are the only 1-block quivers.*

Proof. Suppose that $(Q, \mathcal{B})$ is not a trivial quiver, then by Definition 3.1 $\exists\ a \in Q_1$. If a is a loop then there is no block structure. This implies that $\exists\ U, V \in Q_0$ such that $a \in Q_{1_{U,V}}$. Thus U and V are not in the same block. Hence Q cannot be 1-block. □

The following theorem shows that the vertices of a quiver which are in the same block are indistinguishable.

Theorem 3.15. *There is an equivalence relation on the vertices of* $(Q, \mathcal{B})$ *defined by the blocks.*

Proof. We shall show that $U \sim V$ if $\exists\ B_\ell \in \mathcal{B}$ such that $U, V \in B_\ell$ is an equivalence relation. $U \sim U$ since by Lemma 3.6, $\forall\ U \in Q_0\ \exists!\ B_i \in \mathcal{B}$ such that $U \in B_i$. Thus reflexivity holds. We now show that symmetry holds: $U \sim V \Rightarrow \exists\ B_j \in \mathcal{B}$ such that $U, V \in B_j \Rightarrow V \sim U$. Lastly we show that transitivity holds: $U \sim V \Rightarrow \exists\ B_1 \in \mathcal{B}$ such that $U, V \in B_1$ and $V \sim W \Rightarrow \exists\ B_2 \in \mathcal{B}$ such that $V, W \in B_2$. Since $V \in B_1$ and $V \in B_2$ then $B_1 = B_2$ (because by Lemma 3.6 $\exists!\ B_i \in \mathcal{B}$ such that $V \in B_i$). Then $U, V \in B_i$ and $V, W \in B_i$. This implies that $U, W \in B_i$. Therefore $\exists\ B_i \in \mathcal{B}$ such that $U, W \in B_i \Rightarrow U \sim W$. □

4 Block mutation

Since a block structure of a quiver is not unique, it is almost impossible to define block mutation on such a quiver. We thus associate a block quiver to a quiver with block structure and prove its existence and uniqueness.

Definition 4.1. *Given a quiver with block structure* $(Q, \mathcal{B})$*, we define the* ***block quiver*** $\mathcal{B}Q$ *to be a quiver with vertices, the blocks* B_i*, such that* $\forall\ B_1, B_2 \in \mathcal{B}$*,* $\mathcal{B}Q_{B_1,B_2} \cong Q_{U,V}\ \forall\ U \in B_1, V \in B_2$.

Theorem 4.2. *For any quiver with block structure* $(Q, \mathcal{B})$*, the block quiver* $\mathcal{B}Q$ *exists and is unique.*

Proof. Existence: We shall show that $Q_{U,V} \cong Q_{U',V'}\ \ \forall\ U, U' \in B_1$ and $\forall\ V, V' \in B_2$. Since $B_1 \cap B_2 = \phi$ then $V, V' \notin B_1$ and $U, U' \notin B_2$. By Definition 3.2, $Q_{U,V} \cong Q_{U',V}$ and $Q_{U',V} \cong Q_{U',V'}$. This implies that $Q_{U,V} \cong Q_{U',V'}$. Hence $\mathcal{B}Q$ exists.

Uniqueness: We shall show that if $W \notin B_1, B_2$ then $Q_{U,V} \ncong Q_{U,W}$ and $Q_{U,V} \ncong Q_{W,V}\ \ \forall\ U \in B_1,\ V \in B_2$.

By Definition 3.2,

$$Q_{U',W} \cong Q_{U,W} \text{ for some } U' \in B_1 \tag{1}$$

Now
$$\begin{aligned} Q_{U,V} &\cong Q_{U',V}(\text{by Definition 3.2}) \\ &\cong Q_{U',V'}(\text{by Definition 3.2}) \text{ for some } V' \in B_2 \\ &\ncong Q_{U',W}(\text{since } W \notin B_2) \\ &\cong Q_{U,W}(\text{from (1)}) \end{aligned}$$

Hence $Q_{U,V} \ncong Q_{U,W}$ if $W \notin B_2, \forall\, U \in B_1,\ \ V \in B_2$
Similarly, by Definition 3.2,

$$Q_{V,W} \cong Q_{V',W} \text{ for some } V' \in B_2 \tag{2}$$

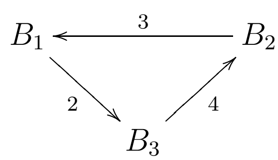
Now
$$\begin{aligned} Q_{U,V} &\cong Q_{U,V'}(\text{by Definition 3.2}) \\ &\cong Q_{U',V'}(\text{by Definition 3.2}) \text{ for some } U' \in B_1 \\ &\ncong Q_{W,V'}(\text{since } W \notin B_1) \\ &\cong Q_{W,V}(\text{from (2)}) \end{aligned}$$

Hence $Q_{U,V} \ncong Q_{W,V}$ if $W \notin B_1, \forall\, U \in B_1,\ \ V \in B_2$. Thus $\mathcal{B}Q$ is unique. □

Since a block quiver is unique for any quiver with block structure, we define block mutation as follows,

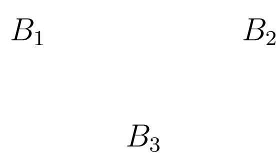
Definition 4.3. *Mutation of a block quiver at a block is called* ***block mutation****.*

Example 4.4. Consider the block quiver,

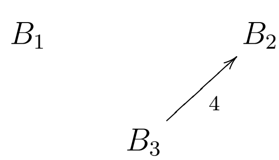


where $B_1 = \{U, V, W, T\}, B_2 = \{P, R, M\}, B_3 = \{X, Y\}$.
Lets do block mutation to this quiver at B_1.

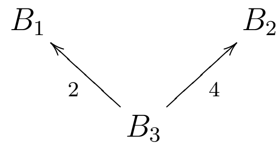
i) The blocks of the quiver remain unchanged.



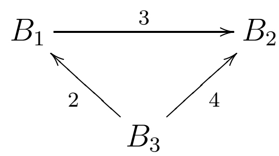
ii) All arrows not incident to B_1 remain unchanged.



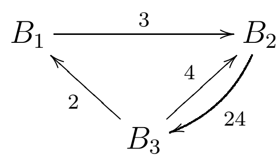
iii) All arrows out of B_1 become arrows into B_1.



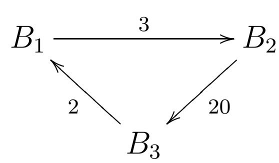
iv) All arrows into B_1 become arrows out of B_1.



v) Add a new arrow $(ba) : B_2 \longrightarrow B_3$ for each pair of arrows $a : B_2 \longrightarrow B_1$ and $b : B_1 \longrightarrow B_3$. Thus there are $2 \times 3 \times 4 = 24$ new arrows.



vi) If there are arrows going both ways between two blocks, final number of arrows is their difference in the direction of the majority arrows i.e. the reduced quiver has $24 - 4 = 20$ arrows between the blocks B_2 and B_3.



In the example above, the number of arrows between the blocks B_2 and B_3 in the reduced quiver is given as $20 \times 3 \times 2 = 120$ (as remarked in Remark 4.6 below) where 3 and 2 are the number of vertices in the blocks B_2 and B_3 respectively.

Notation 4.5. Let B_1 and B_2 be blocks of a block quiver $\mathcal{B}Q$. For all $U, V \in Q_0$, let $A_{U,V} \subset Q_1$ denote the set of all arrows $a_i \in Q_1$ such that $s(a_i) = V$ and $t(a_i) = U$. Also $A_{U,B_1} \subset Q_1$ is the set of all arrows $d_j \in Q_1$ such that $s(d_j) = V$ and $t(d_j) = U$, $\forall\ V \in B_1$. The set of all such arrows from all vertices in B_1 to all vertices in B_2 is denoted by A_{B_2,B_1}. We shall denote the number of elements in a finite set $A_{U,V}$ by $|A_{U,V}|$.

Remark 4.6. *Let $B_1 = \{U_1, ..., U_p\}$, $B_2 = \{V_1, ..., V_q\}$ be blocks of a block quiver $\mathcal{B}Q$. Then $|B_1| = p$ and $|B_2| = q$. Let $m := |A_{U,V}|$ for any $U \in B_1$ and $V \in B_2$. It follows that $|A_{U,B_2}| = mq$, $|A_{B_1,V}| = mp$ and $|A_{B_1,B_2}| = mpq$.*

We now show that mutation of a quiver Q at all the vertices in a block gives the same mutant irrespective of the order of mutations.

Lemma 4.7. *Let $\mathcal{B}Q$ be the block quiver of a quiver Q, with block structure $\mathcal{B} = \{B_1, B_2, B_3\}$ such that $(Q, \mathcal{B})$ has no 2-cycles. Then mutation of Q at all the vertices $U \in B_1$, gives the same mutant irrespective of the order in which mutations are done.*

Proof. Let $B_1 = \{U_1, ..., U_p\}$, $B_2 = \{V_1, ..., V_q\}$ and $B_3 = \{W_1, ..., W_r\}$. By Definition 3.2, $\forall\ U \in B_1$, $A_{U,B_1} = \phi$ and $A_{B_1,U} = \phi$. Since $(Q, \mathcal{B})$ has no 2-cycles then either $A_{U,B_2} = \phi$ or $A_{B_2,U} = \phi$. Similarly, either $A_{U,B_3} = \phi$ or $A_{B_3,U} = \phi$. Let $\tilde{\mu}_U(Q)$ and $\mu_U(Q)$ be the unreduced and reduced mutation of Q respectively. By the steps of mutation at any vertex $U \in B_1$, $\forall\ U_i \neq U$, A_{Y,U_i} and $A_{U_i,Y}$ $\forall\ Y \in Q_0$ remain unchanged. We consider the two cases below:
Case 1. Assume that $\forall\ \ U \in B_1$, $A_{U,B_2} = \phi = A_{U,B_3}$, $A_{B_2,U} = \phi = A_{B_3,U}$, $A_{U,B_2} \neq 0$ and $A_{U,B_3} \neq 0$, or $A_{B_2,U} \neq 0$ and $A_{B_3,U} \neq 0$. This implies that there are no paths of length two through $U \in B_1$ to give a composite arrow when mutating at $U \in B_1$. Hence mutation at any $U_i \in B_1$ only changes arrows into U_i or out of U_i. Thus mutations at multiple U_i give the same mutant irrespective of the order of mutations.
Case 2. Assume that $\forall\ \ U \in B_1$, $A_{U,B_2} \neq \phi$ and $A_{B_3,U} \neq \phi$, or $A_{B_2,U} \neq \phi$ and $A_{U,B_3} \neq \phi$. By relabeling the blocks, the two situations are the same. Thus without loss of generality we only consider $A_{U,B_2} \neq \phi$ and $A_{B_3,U} \neq \phi$. Lets mutate Q at $U \in B_1$. In $\tilde{\mu}_U(Q)$, $|\tilde{\mu}_U(A_{B_3,B_2})| = |A_{B_3,U}||A_{U,B_2}| + |A_{B_3,B_2}|$ and $|\tilde{\mu}_U(A_{B_2,B_3})| = |A_{B_2,B_3}|$. Now if $|\tilde{\mu}_U(A_{B_3,B_2})| - |\tilde{\mu}_U(A_{B_2,B_3})| \geq 0$ then $|\mu_U(A_{B_3,B_2})| = |\tilde{\mu}_U(A_{B_3,B_2})| - |\tilde{\mu}_U(A_{B_2,B_3})|$. But if $|\tilde{\mu}_U(A_{B_2,B_3})| - |\tilde{\mu}_U(A_{B_3,B_2})| \geq 0$, then $|\mu_U(A_{B_2,B_3})| = |\tilde{\mu}_U(A_{B_2,B_3})| - |\tilde{\mu}_U(A_{B_3,B_2})|$.
Since these numbers of arrows are the same with mutation at any vertex $U \in B_1$, then these numbers do not depend on the vertices at which mutations are done but on the number of mutations. □

In the next theorem, we show that block mutation gives the same mutant as mutating the quiver at all the individual vertices in the block. This is in line with the findings of Karpov and Nogin [1] in the context of exceptional collections.

Theorem 4.8. *Let $\mathcal{B}Q$ be the block quiver of a quiver Q, with block structure $\mathcal{B} = \{B_1, B_2, B_3\}$ such that $(Q, \mathcal{B})$ has no 2-cycles. Then mutation of $\mathcal{B}Q$ at a block B_1 gives the same mutant as mutating the quiver Q at all the vertices $U \in B_1$.*

Proof. Let $B_1 = \{U_1, ..., U_p\}$, $B_2 = \{V_1, ..., V_q\}$ and $B_3 = \{W_1, ..., W_r\}$. Since $(Q, \mathcal{B})$ has no 2-cycles then either $A_{B_1,B_2} = \phi$ or $A_{B_2,B_1} = \phi$. Similarly, either $A_{B_1,B_3} = \phi$ or $A_{B_3,B_1} = \phi$. Let $\tilde{\mu}_{B_1}(\mathcal{B}Q)$ and $\mu_{B_1}(\mathcal{B}Q)$ be the unreduced and reduced mutation of $\mathcal{B}Q$ respectively. We consider the two cases below:

Case 1. Assume that $A_{B_1,B_2} = \phi = A_{B_1,B_3}$, $A_{B_2,B_1} = \phi = A_{B_3,B_1}$, $A_{B_1,B_2} \neq 0$ and $A_{B_1,B_3} \neq 0$, or $A_{B_2,B_1} \neq 0$ and $A_{B_3,B_1} \neq 0$. This implies that there are no paths of length two through $B_1 \in \mathcal{B}$ to give a composite arrow when mutating at $B_1 \in \mathcal{B}$. Hence mutation at B_1 only changes arrows into B_1 or out of B_1. Also mutation at any $U_i \in B_1$ only changes arrows into U_i or out of U_i. Thus mutation of $\mathcal{B}Q$ at a block B_1 gives the same mutant as mutating the quiver Q at all the vertices $U \in B_1$.

Case 2. Assume that $A_{B_1,B_2} \neq \phi$ and $A_{B_3,B_1} \neq \phi$, or $A_{B_2,B_1} \neq \phi$ and $A_{B_1,B_3} \neq \phi$. By relabeling the blocks, the two situations are the same. Thus without loss of generality we only consider $A_{B_1,B_2} \neq \phi$ and $A_{B_3,B_1} \neq \phi$. Let $m := |A_{U,V}|$ and $n := |A_{W,U}|$ for all $U \in B_1$, $V \in B_2$, $W \in B_3$ then $|A_{U,B_2}| = mq$, $|A_{B_3,U}| = nr$. Let $l := |A_{B_3,B_2}|$ and $k := |A_{B_2,B_3}|$.

Lets mutate Q at $B_1 \in \mathcal{B}$. In $\tilde{\mu}_{B_1}(Q)$, $|\tilde{\mu}_{B_1}(A_{B_3,B_2})| = |A_{B_3,B_1}||A_{B_1,B_2}| + |A_{B_3,B_2}|$ and $|\tilde{\mu}_{B_1}(A_{B_2,B_3})| = |A_{B_2,B_3}|$. Now if $|\tilde{\mu}_{B_1}(A_{B_3,B_2})| - |\tilde{\mu}_{B_1}(A_{B_2,B_3})| = mnpqr + l - k \geq 0$ then $|\mu_{B_1}(A_{B_3,B_2})| = mnpqr + l - k$. But if $|\tilde{\mu}_{B_1}(A_{B_2,B_3})| - |\tilde{\mu}_{B_1}(A_{B_3,B_2})| = k - l - mnpqr \geq 0$, then $|\mu_{B_1}(A_{B_2,B_3})| = k - l - mnpqr$.

Lets mutate Q at vertex $U_1 \in B_1$. Now $|\tilde{\mu}_{U_1}(A_{B_3,B_2})| = |A_{B_3,U_1}||A_{U_1,B_2}| + |A_{B_3,B_2}|$ and $|\tilde{\mu}_{U_1}(A_{B_2,B_3})| = |A_{B_2,B_3}|$. If $|\tilde{\mu}_{U_1}(A_{B_3,B_2})| - |\tilde{\mu}_{U_1}(A_{B_2,B_3})| = mnqr + l - k \geq 0$ then $|\mu_{U_1}(A_{B_3,B_2})| = mnqr + l - k$. But if $|\tilde{\mu}_{U_1}(A_{B_2,B_3})| - |\tilde{\mu}_{U_1}(A_{B_3,B_2})| = k - l - mnqr \geq 0$, then $|\mu_{U_1}(A_{B_2,B_3})| = k - l - mnqr$. Lets mutate $\mu_{U_1}(Q)$ at vertex $U_2 \in B_1$. Then $|\tilde{\mu}_{U_2}\mu_{U_1}(A_{B_3,B_2})| = |A_{B_3,U_2}||A_{U_2,B_2}| + |\mu_{U_1}(A_{B_3,B_2})|$ and $|\tilde{\mu}_{U_2}\mu_{U_1}(A_{B_2,B_3})| = |\mu_{U_1}(A_{B_2,B_3})|$. If $|\tilde{\mu}_{U_2}\mu_{U_1}(A_{B_3,B_2})| - |\tilde{\mu}_{U_2}\mu_{U_1}(A_{B_2,B_3})| = 2mnqr + l - k \geq 0$ then $|\mu_{U_2}\mu_{U_1}(A_{B_3,B_2})| = 2mnqr + l - k$. But if $|\tilde{\mu}_{U_2}\mu_{U_1}(A_{B_2,B_3})| - |\tilde{\mu}_{U_2}\mu_{U_1}(A_{B_3,B_2})| = k - l - 2mnqr \geq 0$, then $|\mu_{U_2}\mu_{U_1}(A_{B_2,B_3})| = k - l - 2mnqr$.

Now, if mutations are continued until the quiver is mutated at all the p vertices in the block B_1 then either $|\mu_{U_p}...\mu_{U_1}(A_{B_3,B_2})| = mnpqr + l - k$ or $|\mu_{U_p}...\mu_{U_1}(A_{B_2 B_3})| = k - l - mnpqr$. These numbers match up the number of arrows obtained as a result of mutation of the block quiver. By Lemma 4.7, the number of arrows in $\mu_{U_p}...\mu_{U_1}(Q)$ due to all permutations of $\{U_1, ..., U_p\}$ match up the number of arrows obtained as a result of mutation of the block quiver. □

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