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A Note on Nilpotent Lattice Matrices

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Abstract

Some properties and characterizations for nilpotent matrices are established and in particular, a necessary and sufficient condition for an $n \times n$ nilpotent matrix to have the nilpotent index 2 and 3 is given.

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1 Introduction

A matrix is called a lattice matrix if its entries belong to a distributive lattice. A square lattice matrix A is called nilpotent if $A^m = 0$ for some positive integer m, where 0 is the zero matrix. It is clear that any Boolean matrix and fuzzy matrix are lattice matrices. Powers of real matrices computed with at least one of the addition/multiplication operations replaced by maximum or minimum have been extensively studied, since they have applications in various types of discrete systems. In particular, for the description of the behavior of fuzzy systems, whose states are expressed using fuzzy values taken from the real interval $(0, 1)$ or an arbitrary distributive lattice, the powers of lattice matrices are of special importance.

Lattice matrices are useful tools in various domains like the theory of switching nets, automata theory and the theory of finite graphs. The notion of nilpotent lattice matrices seems to have appeared first in the work of Give'on [5]. In [5], Give'on proved that an $n \times n$ lattice matrix A is nilpotent if and only if $A^n = O$. Since then, a number of researchers have studied the topic of nilpotent lattice matrices. In [4] K. Cechlárová stressed the significance of graph-theoretical approach in the study of such matrices. Y.J.Tan has improved the earlier works in [2], [3]. In [2], Y.J.Tan has studied also the nilpotent index of nilpotent lattice matrices and obtained a necessary and sufficient condition for $n \times n$ lattice matrices, whose nilpotent index is equal to n.

In this paper, we continue to discuss the topic of nilpotent lattice matrices. We give a necessary and sufficient condition for a nilpotent matrix over a $D_{01}-$lattice, which has the nilpotent index $2, 3$.

2 Definitions and preliminary lemmas

A lattice is a poset $(L, \leq)$, in which every two elements $a, b \in L$ have a unique least upper bound and a unique greatest lower bound. We denote them respectively by $a \vee b$ and $a \wedge b$. We use also $a + b$ and $a.b$ instead of $a \vee b$ and $a \wedge b$ respectively. A lattice $(L, \leq, +, .)$ is said to be distributive if $+$ and $.$ are distributive with respect to each other. A $D_{01}-$lattices is a lattice L , in which $0, 1 \in L$ and for each $x \in L$ we have $0 \leq x \leq 1$. The elements $0, 1 \in L$ are called respectively universal lower bound and universal upper bound.

The Fuzzy algebra $[0, 1]$ is a $D_{01}-$lattices with respect to operations "max" and "min". The least element of the set is called the relative lower pseudo complement of b in a, and we denote it by $a - b$. If for any pair of elements $a, b \in L$, $a - b$ exists, then L is said to be a dually Brouwerian lattice. Trivially the Fuzzy algebra is a dually Brouwerian lattice:

$$a - b = \begin{cases} a & if\, b < a \\ 0 & if\, a \leq b. \end{cases}$$

Let $(L, \leq, +, .)$ be a distributive lattice with $0, 1$. We denote by $M_n(L)$, the set of all $n \times n (n \geq 2)$ matrices over L and denote by a_{ij}, the $(i, j) - th$ entry of the lattice matrix $A \in M_n(L)$. Now let $A, B, C \in M_n(L)$ and let $N = \{1, ..., n\}$, we define:

$A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for $i, j \in N$,
$A + B = C$ if and only if $a_{ij} + b_{ij} = c_{ij}$ for $i, j \in N$,
$A \wedge B = C$ if and only if $a_{ij} \wedge b_{ij} = c_{ij}$ for $i, j \in N$,
$AB = C$ if and only if $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ for $i, j \in N$,
$I_n = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & if\, i = j \\ 0 & if\, i \neq j \end{cases} , i, j \in N.$$

The following properties are derived immediately from these definitions:

Lemma 2.1.([3]) For $A, B, C, D \in M_n(L)$ we have:
(1) $(A + B) + C = A + (B + C), (AB)C = A(BC)$,
(2) $A + B = B + A$,
(3) $A + A = A$,
(4) $A \wedge A = A$,
(5) if $A \leq B, C \leq D$ then $AC \leq BD$.

For $A \in M_n(L)$, if $A^2 \leq A$, then A is called *transitive*; if $a_{ii} = 0$ for all $i \in N$, then A is called *irreflexive*; if for all $i, j \in N$, $a_{ij} = 0$, then A is called the zero matrix. The *zero matrix* is denoted by O. A is called *nilpotent*, if $A^m = O$ for some integer m. The least integer satisfying $A^m = O$, is called the *nilpotent index* of A and is denoted by $h(A)$. The *determinant* $det(A)$ of a lattice matrix A defined as follows:

$$det(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} ... a_{n\sigma(n)},$$

where S_n denotes the symmetric group of the set $\{1, 2, ..., n\}$.

For $A \in M_n(L)$, we shall denote by $A(i_1, i_2, ..., i_t \mid j_1, j_2, ..., j_t)$ the $(n - t) \times (n - t)$ submatrix of A obtained from A by eliminating rows $i_1, i_2, ..., i_t$ and columns $j_1, j_2, ..., j_t$. Then $A(i_1, i_2, ..., i_t \mid i_1, i_2, ..., i_t)$ is called a *principal submatrix* of A of order $n - t$ and $detA(i_1, i_2, ..., i_t \mid i_1, i_2, ..., i_t)$ is called a *principal minor* of A of order $n - t$. Also $adjA = (perA(i \mid j))^T_{n \times n}$, $adj(A)$ is called the *adjoint matrix* of A.

The following lemmas are used in the sequel.

Lemma 2.2.([2]) Let $A = (a_{ij}) \in M_n(L)$, then $det(A) = \sum_{i=1}^{n} a_{ij} detA(i \mid j)$, $j = 1, 2, ..., n$.

For a lattice L, the element $a \in L$ is called an *atom* of L if $a > 0$ and for every $b \in L$ with $a \geq b > 0$, we have $a = b$.

Lemma 2.3.([1]) Every finite distributive lattice can be embedded into a finite Boolean lattice.

By lemma 2.3 the sublattice $L(A)$, $A \in M_n(L)$, generated by $k(A)$, the set of all different entries in A, can be embedded into a finite Boolean lattice. We denote this Boolean lattice by $\beta(A)$, the Boolean lattice associated with A. We denote also the set of atoms of $\beta(A)$ by $\sigma(A)$.

Lemma 2.4.([4]) A matrix $A \in M_n(L)$ can be represented by binary matrices A_s, $s \in \sigma(A)$ of the form:

$$(A_s)_{ij} = \begin{cases} 1 & a_{ij} \geq s \\ 0 & otherwise, \end{cases}$$

called the $s-th$ *constituent* of A. Now A can be uniquely expressed as a linear combination of its constituents with coefficients s, $s \in \sigma(A)$, in the following way:

$$A = \sum_{s\in\sigma(A)} sA_s.$$

We have also for each ν; $A^\nu = \sum_{s\in\sigma(A)} sA_s^\nu$.

Now we review basic notions and facts of graph theory used in this paper. A *digraph* is a pair $G = (V, H)$, where V is a finite set, called the vertex set, and H is a subset of $V \times V$, called the *arc set*. A sequence of vertices $p = (i_0, i_1, ..., i_m)$ is called a *path* from vertex i_0 to vertex i_m, if for all $j = 1, 2, ..., m$ the pair $(i_{j-1}, i_j) \in H$; its length is m and is denoted by $l(p)$. A path is called a *cycle* if $i_0 = i_m$, a cycle of length 1 is called a *loop*. A digraph that does not contain any cycle is called *asyclic*.
Now we define the *associated digraph* $G(A) = (V, H)$ of a binary matrix A of order n to be the digraph on the set of vertices $V = \{1, 2, ..., n\}$ such that:

$$(i, j) \in H \text{ if and only if } a_{ij} = 1.$$

There is a well-known connection between the entries in powers of binary matrices and paths in associated digraph:
$(i, j) - th$ entry $a_{ij}^{(k)}$ in A^k is equal to 1 iff in the associated digraph $G(A)$ of A there exist a path of length k from vertex i to vertex j.

We now mention the following result of K. Cechlárová [4], which will be used in the sequel:

Theorem 2.5.([4]) Let A be a binary matrix of order n and G its associated digraph, then A is nilpotent if and only if G is acyclic.

3 Nilpotent index of matrices over D_{01}-lattices

Yi-jia Tan in [2] obtained some properties and characterizations for nilpotent matrices and established a necessary and sufficient condition for an $n \times n$ nilpotent matrix to have the nilpotent index n. Following Tan's result we provide a partial answer for Tan's question and provide a necessary and sufficient condition for an $n \times n$ nilpotent matrix to have the nilpotent index 2 and 3.

We start by providing the following results which will be needed in the proof of our main results:

Lemma 3.1.([2]) Let $A \in M_n(L)$. Then A is nilpotent if and only if every principal minor of A is 0.

By Lemma 2.2 and 3.1 we have:

Corollary 3.2.([2]) If $A \in M_n(L)$ is nilpotent, then $detA = 0$.

We now observe that the converse of this corollary is also true:

Proposition 3.3. If $detA = 0$, then A is nilpotent.

Proof. Suppose that $detA = 0$. According to the definition of $detA$, the associated digraph of the constituents of A must be acyclic. Hence A does not have any cycle. By Theorem 2.5, it implies that A is nilpotent.

Proposition 3.4.([2]) (Give'on) For $A \in M_n(L)$, A is nilpotent if and only if $A^n = 0$.

Proposition 3.5.([2]) Let $A \in M_n(L)$, then A is nilpotent if and only if $a_{ii}^{(k)} = 0$, where $i, k \in N$ and $A^k = (a_{ij}^{(k)})$.

Proposition 3.6.([2]) Let $A \in M_n(L)$. If A is transitive and irreflexive, then A is nilpotent.

We now show that the converse of this result is also true:

Proposition 3.7. If A is nilpotent, then A is irreflexive and transitive.

Proof. Suppose that A is nilpotent. If A is not transitive, then $A \neq 0$. We now consider the following:

Case$I$: $A^2 = A$, then $A = A^2 = A^3 = ... = A^n$ and so $A^n = A \neq 0$. A contradiction to the assumption that A is nilpotent.

CaseII: $A^2 \geq A$, then by Lemma 2.1(5) we have $A \leq A^2 \leq A^3 \leq ... \leq A^n$ and therefore $A^n \geq A \neq 0$, which is also a contradiction.

Now suppose that A is not irreflexive. Then $a_{ii} \neq 0$, for some $i \in N$. By Proposition 3.2, we deduce that A is not nilpotent, which is also a contradiction.

Proposition 3.8.([2]) If $A \in M_n(L)$ is nilpotent, then

(1) $AadjA = 0$ and $(adjA)A = 0$,

(2) $(adjA)^2 = 0$.

Corollary 3.9. If $A \in M_n(L)$ is nilpotent, then $adjA$ is transitive and irreflexive.

Proof. By Proposition 3.8, $adjA$ is nilpotent. Now Proposition 3.7 implies that $adjA$ is transitive and irreflexive.

Now we will discuss the nilpotent index of a lattice matrix A. Y.J. Tan, in [2], provided a necessary and sufficient condition for nilpotent index of a matrix A of order n, to be n:

Proposition 3.10.([2]) Let $A \in M_n(L)$ be nilpotent. Then $h(A) = n$ if and only if $adjA \neq O$.

Y.J. Tan, in [2] raised an open problem, that:

How to describe the nilpotent lattice matrices with any given nilpotent index r?

We now provide a partial answer to the problem and give a necessary and sufficient condition for a lattice matrix over D_{01}-lattices to have nilpotent index 2 and 3:

Theorem 3.11. Let L be a D_{01}-lattice. If $A \in M_n(L)$ is nilpotent, then $h(A) = 2$ if and only if $A \neq 0$, and for every $i, j \in N$, $R_i \wedge R_j^T = 0$, where R_k is a nonzero row of A.

Proof. Suppose first that $h(A) = 2$, then $A \neq 0$. Now we show that for every $i, j \in N$, $R_i \wedge R_j^T = 0$. If there exist p, q such that $R_p \wedge R_q^T \neq 0$, then there exists s such that $a_{ps}a_{sq} > 0$ and so $\sum_{m=1}^{n} a_{pm}a_{mq} \geq a_{ps}a_{sq} > 0$. This is a contradiction, since $h(A) = 2$.

Conversely, let $A \neq 0$. The entries of A^2 have the form $\sum_{r=1}^{n} a_{ir}a_{rj}$. But for every i, j we have $R_i \wedge R_j^T = 0$, therefore the terms of this sum are all 0, and hence $A^2 = 0$.

Theorem 3.12. Let L be a D_{01}-lattice and assume that $A \in M_n(L)$ is nilpotent. Then $h(A) = 3$ iff $R_i \wedge R_j^T \neq 0$ for some i, j, and $A \wedge A^T = 0$.

Proof. Suppose first that $h(A) = 3$. Since $A^2 \neq 0$ then by Theorem 3.11, there must exist s, t such that $R_s \wedge R_t^T \neq 0$. Now we show that $A \wedge A^T = 0$. Suppose that $A \wedge A^T \neq 0$, then we can find p, q such that $a_{pq}a_{qp} > 0$. Therefore $a_{qp}a_{pq}a_{qp} > 0$, which is a term of (p, q)-th entry of the matrix A^3. So

$$\sum_{1\le i_1,i_2\le n} a_{qi_1}a_{i_1i_2}a_{i_2p} > a_{qp}a_{pq}a_{qp} > 0,$$

which leads to a contradiction, since $A^3 = 0$.

Conversely, by Theorem 3.11, since $R_i \wedge R_j^T \neq 0$ for some i, j, we have $A^2 \neq 0$. From $A \wedge A^T = 0$ we now get $A^3 = 0$.

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