International Journal of Algebra, Vol. 5, 2011, no. 2, 71 - 81

Weak Armendariz Skew Polynomial Rings

A. Mousavi[a], F. Keshavarz[b], M. Rasuli[c] and A. Alhevaz[d]

[a] Department of Mathematics, Islamic Azad University,
South Tehran Branch, Tehran, Iran.
[b] Member of Young Researchers Club, Islamic Azad University,
North Tehran Branch, Tehran, Iran.
[c] Department of Mathematics, Isfahan University of Technology,
P.O.Box: 84156-83111, Isfahan, Iran.
[d] Department of Pure Mathematics, Faculty of Mathematical Sciences,
Tarbiat Modares University, Tehran, Iran, P.O. Box 14115-134.
fatemeh.keshavarz.2003@gmail.com.
a.alhevaz@yahoo.com. m.rasuli@yahoo.com.
Corresponding author. a_mousavi@azad.ac.ir

Abstract

A ring R is called weak Armendariz if whenever the product of any two polynomials in $R[x]$ over R is zero, then the product of any pair of coefficients from the two polynomials be a nilpotent element of R. For a ring endomorphism α, we introduce the notion of α-weak Armendariz rings by considering the polynomials in the skew polynomial ring $R[x;\alpha]$ instead of $R[x]$, which are a generalization of weak Armendariz rings and α-Armendariz rings. Basic properties of α-weak Armendariz rings are observed, and connections of properties of an α-weak Armendariz ring R with those of $R[x;\alpha]$ are investigated. As a consequence our results improve not only some results of Liu [8], but also some known results on Armendariz rings.

Mathematics Subject Classification: $16S36$; $16D25$.

Keywords: weak Armendariz ring, semicommutative ring, α-compatible ring, skew polynomial ring.

Throughout this paper R denotes an associative ring with unity. For a ring R, we denote by $ni\ell(R)$ the set of all nilpotent elements of R and by $R[x]$ the polynomial ring with an indeterminate x over R.

By [11], a ring R is called an *Armendariz* ring if whenever $f(x)g(x) = 0$ where $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$, then $a_ib_j = 0$ for each i, j. The name Armendariz ring is chosen because Armendariz

[2] had noted that a *reduced* ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. Recall that a ring R is called *reversible* if $ab = 0$ implies $ba = 0$, for all $a, b \in R$; R is called *semicommutative* if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Reduced rings are clearly reversible and reversible rings are semicommutative, but the converse is not true in general[10].

Liu and Zhao [8] have studied a generalization of Armendariz rings, which they called weak Armendariz rings. A ring R is called a *weak-Armendariz* ring if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in ni\ell(R)$ for each i, j. Each semicommutative ring is weak Armendariz by [8] and so weak Armendariz rings are a common generalization of semicommutative rings and Armendariz rings.

The Armendariz property of a ring was extended to one of skew polynomials [4,5]. For an endomorphism α of a ring R, the *skew polynomial* ring $R[x; \alpha]$ consists of the polynomials in x with coefficients in R written on the left, subject to the relation $xr = \alpha(r)x$ for all $r \in R$. A ring R is called *α-Armendariz* (resp., *α-skew Armendariz*) if for $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ (resp., $a_i\alpha^i(b_j)$=0) for all $0 \le i \le m$ and $0 \le j \le n$ [5, Definition 1.1](resp., [4, Definition]). According to Hashemi and Moussavi [3], a ring R is said to be *α-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. For α-compatible rings the above mentioned concepts are equivalent. According to Krempa [6], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is called *α-rigid* if there exist a rigid endomorphism α of R. By [3, Lemma 2.2], a ring R is α-rigid if and only if R is α-compatible and reduced. Also by [4, Proposition 3], if R is α-rigid ring, then $R[x; \alpha]$ is reduced.

We are motivated to introduce the notion of α-weak Armendariz ring R with respect to an endomorphism α of R. This notion extends both ring weak Armendariz rings and α-Armendariz rings. We do this by considering the weak Armendariz condition on polynomials in $R[x; \alpha]$ instead of $R[x]$. This provides us with an opportunity to study weak Armendariz rings in a general setting, and several known results on weak Armendariz rings are obtained as corollaries. We start with the following definition.

Definition 1. Let α be an endomorphism of a ring R. The ring R is called *α-weak Armendariz* (resp., *α-skew weak Armendariz*) if for $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $a_ib_j \in ni\ell(R)$ (resp., $a_i\alpha^i(b_j) \in ni\ell(R)$) for all $0 \le i \le m$ and $0 \le j \le n$.

It is clear that a ring R is weak Armendariz if R is id_R-weak Armendariz, where id_R is the identity endomorphism of R.

Proposition 2. *For an endomorphism α of a ring R, we have the following statements:*
(1) *If R is α-Armendariz (resp., α-skew Armendariz) ring, then R is α-weak Armendariz (resp., α-skew weak Armendariz).*
(2) *Every subring S with $\alpha(S) \subseteq S$ of an α-weak Armendariz ring is also α-weak Armendariz.*
Proof. It is clear by definition.

The following example shows that there exists an endomorphism α of a weak Armendariz ring R such that R is not α-weak Armendariz.

Example 3. Let $R = R_1 \oplus R_2$, where R_1 and R_2 be any reduced rings. Then R is a semicommutative and so R is weak Armendariz. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. Let $f(x) = (0, 1) - (0, 1)x$, $g(x) = (1, 0) + (0, 1)x \in R[x; \alpha]$. Then $f(x)g(x) = 0$, but $(0, 1)(0, 1) = (0, 1) \notin ni\ell(R)$. Therefore R is neither α-weak Armendariz nor α-skew weak Armendariz.

Any endomorphism α of R can be extended to an endomorphism $\bar{\alpha}$ of $T_n(R)$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$.

Proposition 4. *Let α be an endomorphism of a ring R. Then R is α-weak Armendariz if and only if, for any n, $T_n(R)$ is $\bar{\alpha}$-weak Armendariz.*

Proof. We only prove necessity, since subrings of α-weak Armendariz rings is also α-weak Armendariz. Note that $T_n(R)[x; \bar{\alpha}] \cong T_n(R[x; \alpha])$. Let $f(x) = \sum_{i=0}^{p} A_i x^i$ and $g(x) = \sum_{j=0}^{q} B_j x^j \in T_n(R)[x; \bar{\alpha}]$ are such that $f(x)g(x) = 0$. Let

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix},\ B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b_{22}^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^j \end{pmatrix},$$

for $0 \leq i \leq p$, $0 \leq j \leq q$. Then $f_s(x) = \sum_{i=0}^{p} a_{ss}^i x^i$ and $g_s(x) = \sum_{j=0}^{q} b_{ss}^j x^j \in R[x; \alpha]$ and $f_s(x)g_s(x) = 0$, for each $1 \leq s \leq n$. Since R is α-weak Armendariz ring, there exists $m_{ijs} \in \mathbb{N}$ such that $(a_{ss}^i b_{ss}^j)^{m_{ijs}} = 0$, for any s and any i, j. Let $m_{ij} = max\{m_{ijs} \mid 1 \leq s \leq n\}$. Then $[(A_i B_j)^{m_{ij}}]^n = 0$. Therefore $T_n(R)$ is $\bar{\alpha}$-weak Armendariz.

Corollary 5. [8, Proposition 2.2] *A ring R is a weak Armendariz ring if and only if, for any n, $T_n(R)$ is weak Armendariz.*

Definition 6. For a ring R, consider the following set of triangular matrices:

$$T(R,n) := \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \ddots & a_{n-2} \\ 0 & 0 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_1 \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R \right\}, \text{ with } n \geq 2.$$

It is easy to see that $T(R,n)$ is a subring of the triangular matrix rings, with matrix addition and multiplication. We can denote elements of $T(R,n)$ by $(a_0, a_1, \ldots, a_{n-1})$, then $T(R,n)$ is a ring with addition point-wise and multiplication given by
$(a_0, a_1, \ldots, a_{n-1})(b_0, b_1, \ldots, b_{n-1}) = (a_0b_0, a_0b_1+a_1b_0, \ldots, a_0b_{n-1}+\cdots+a_{n-1}b_0)$, for each $a_i, b_j \in R$.
On the other hand, there is a ring isomorphism $\varphi : R[x]/\langle x^n\rangle \to T(R,n)$, given by, $\varphi(a_0+a_1x+\cdots+a_{n-1}x^{n-1}) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, $0 \leq i \leq n-1$. So $T(R,n) \cong R[x]/\langle x^n\rangle$, where $R[x]$ is the ring of polynomial in an indeterminate x and $\langle x^n\rangle$ is the ideal generated by x.

Theorem 7. *A ring R is α-weak Armendariz ring if and only if, for any n, $\frac{R[x]}{\langle x^n\rangle}$ is $\bar{\alpha}$-weak Armendariz.*

Proof. We only prove necessity, since subrings of α-weak Armendariz rings is also α-weak Armendariz. Note that $ni\ell(T(R,n)) = (ni\ell(R), R, \ldots, R)$ and also $T(R,n)[x;\bar{\alpha}] \cong T(R[x;\alpha], n)$. Let $F(x) = \sum_{i=0}^{m} A_i x^i$, $G(x) = \sum_{j=0}^{n} B_j x^j$ $\in T(R,n)[x;\bar{\alpha}]$ are such that $F(x)G(x) = 0$. Let

$$A_i = \begin{pmatrix} a_{i0} & a_{i1} & \cdots & a_{i(n-2)} & a_{i(n-1)} \\ 0 & a_{i0} & a_{i1} & \cdots & a_{i(n-2)} \\ 0 & 0 & a_{i0} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{i1} \\ 0 & 0 & \cdots & 0 & a_{i0} \end{pmatrix}, B_j = \begin{pmatrix} b_{j0} & b_{j1} & \cdots & b_{j(n-2)} & b_{j(n-1)} \\ 0 & b_{j0} & b_{j1} & \cdots & b_{j(n-2)} \\ 0 & 0 & b_{j0} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b_{j1} \\ 0 & 0 & \cdots & 0 & b_{j0} \end{pmatrix}$$

for $0 \leq i \leq m$, $0 \leq j \leq n$. Then $f_0(x) = \sum_{i=0}^{m} a_{i0}x^i$ and $g_0(x) = \sum_{j=0}^{n} b_{j0}x^j \in R[x;\alpha]$ and $f_0(x)g_0(x) = 0$. Since R is α-weak Armendariz ring, there exists $m_{ij} \in \mathbb{N}$ such that $(a_{i0}b_{j0})^{m_{ij}} = 0$, for any i, j. Then $A_iB_j \in ni\ell(T(R,n))$. Therefore $\frac{R[x]}{\langle x^n\rangle}$ is $\bar{\alpha}$-weak Armendariz.

If we take $\alpha = id_R$, we deduce:
Corollary 8. A ring R is weak Armendariz if and only if, for any n, $\frac{R[x]}{\langle x^n\rangle}$ is weak Armendariz.

Corollary 9. [8, Theorem 3.9] *If R is a semicommutative ring, then for any*

n, $\frac{R[x]}{\langle x^n \rangle}$ is weak Armendariz.

Recall that for a ring R and an (R,R)-bimodule M, the *trivial extension* of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the multiplication $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 10. *Let α be an endomorphism of a ring R. Then R is α-weak Armendariz if and only if, $T(R,R)$ is $\bar{\alpha}$-weak Armendariz.*

Proof. Observe that $\frac{R[x]}{\langle x^2 \rangle} \cong T(R,R)$.

Example 11. Consider the ring $R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$, where R is an α-rigid ring. The endomorphism α of R is extended to the endomorphism $\bar{\alpha} : R_4 \to R_4$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. The ring R_4 is not $\bar{\alpha}$-Armendariz by [4, Example 18] and [5, Theorem 1.8]. But since R is α-rigid, R is α-weak Armendariz and hence R_4 is $\bar{\alpha}$-weak Armendariz by Propositions 4 and 2(2).

Lemma 12. [3, Lemma 3.2] *Let R be an α-compatible ring. Then we have the following:*

(*i*) If $ab = 0$, then $a\alpha^n(b) = \alpha^n(a)b = 0$, for all positive integers n.

(*ii*) If $\alpha^k(a)b = 0$ for some positive integer k, then $ab = 0$.

We now state another lemma that will be helpful to us in the sequel.

Lemma 13. *Let R be an α-compatible ring. Then we have the following:*

(*i*) If $ab \in ni\ell(R)$, then $a\alpha^m(b) \in ni\ell(R)$ and $\alpha^n(a)b \in ni\ell(R)$, for all positive integers m, n.

(*ii*) If $\alpha^k(a)b \in ni\ell(R)$ or $a\alpha^l(b) \in ni\ell(R)$ for some positive integers k, l, then $ab \in ni\ell(R)$.

Proof. Since $ab \in ni\ell(R)$, there exist some positive integer k such that $(ab)^k = ab\ ab \cdots ab = 0$. Using α-compatibility of R, we have:

$ababab \cdots abab = 0$

$\Leftrightarrow ababab \cdots aba\alpha^m(b) = 0$

$\Leftrightarrow ababab \cdots a\alpha^m(ba\alpha^m(b)) = 0$

$\Leftrightarrow ababab \cdots a\alpha^m(b)\alpha^m(a\alpha^m(b)) = 0$
$\Leftrightarrow ababab \cdots a\alpha^m(b)a\alpha^m(b) = 0$
$\vdots$
$\Leftrightarrow [a\alpha^m(b)]^k = 0$. Therefore $a\alpha^m(b) \in ni\ell(R)$. By a similar way, one can prove the other cases.

Lemma 14. *Let R be α-compatible ring. Then $a^{k_1+k_2+\cdots+k_n} = 0$, implies that $(\alpha^{v_1}(a))^{k_1}(\alpha^{v_2}(a))^{k_2} \cdots (\alpha^{v_n}(a))^{k_n} = 0$, where v_p $(1 \leq p \leq n)$ is a nonnegative integer.*

Proof. Using Lemma 12 and α-compatibility of R, we have:
$a^{k_1+k_2+\cdots+k_n} = 0$
$\Rightarrow a^{k_1}a^{k_2} \cdots a^{k_n} = 0$
$\Rightarrow a^{k_1}a^{k_2} \cdots a^{k_{n-1}}\alpha^{v_n}(a^{k_n}) = 0$
$\Rightarrow a^{k_1}a^{k_2} \cdots a^{k_{n-1}}(\alpha^{v_n}(a))^{k_n} = 0$
$\Rightarrow \alpha^{v_{n-1}}(a^{k_1}a^{k_2} \cdots a^{k_{n-1}})(\alpha^{v_n}(a))^{k_n} = 0$
$\Rightarrow \alpha^{v_{n-1}}(a^{k_1}a^{k_2} \cdots a^{k_{n-2}})\alpha^{v_{n-1}}(a^{k_{n-1}})(\alpha^{v_n}(a))^{k_n} = 0$
$\Rightarrow \alpha^{v_{n-1}}(a^{k_1}a^{k_2} \cdots a^{k_{n-2}})(\alpha^{v_{n-1}}(a))^{k_{n-1}}(\alpha^{v_n}(a))^{k_n} = 0$
$\Rightarrow a^{k_1}a^{k_2} \cdots a^{k_{n-2}}(\alpha^{v_{n-1}}(a))^{k_{n-1}}(\alpha^{v_n}(a))^{k_n} = 0$
$\vdots$
$\Rightarrow (\alpha^{v_1}(a))^{k_1}(\alpha^{v_2}(a))^{k_2} \cdots (\alpha^{v_n}(a))^{k_n} = 0.$

Recall that an ideal I of a ring R is called α-invariant if $\alpha(I) \subseteq I$. If an ideal I of R is α-invariant, then $I[x;\alpha]$ is an ideal of $R[x;\alpha]$, as for any $a \in I$, $\alpha^j(a) \in I$ for all positive integers j.

Corollary 15. *Let R be α-compatible ring. Then $ni\ell(R)[x;\alpha]$ is an ideal of $R[x;\alpha]$.*

Proposition 16. *Let R be semicommutative and α-compatible ring and $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha]$. Then $f(x) \in ni\ell(R[x;\alpha])$ if and only if $a_i \in ni\ell(R)$ for all $0 \leq i \leq n$.*

Proof. Suppose $f(x) = a_0+a_1x+\cdots+a_nx^n \in R[x;\alpha]$ and there exist some positive integer k such that $[f(x)]^k = 0$. Then $a_n\alpha^n(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$, since it is the leading coefficient of $[f(x)]^k$. Using Lemma 11 and α-compatibility of R, we have:
$a_n\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$
$\Rightarrow \alpha^n(a_n)\alpha^n(a_n)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$
$\Rightarrow \alpha^n(a_n^2)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$
$\Rightarrow a_n^2\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$
$\Rightarrow \alpha^{2n}(a_n^2)\alpha^{2n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$
$\Rightarrow \alpha^{2n}(a_n^3)\alpha^{3n}(a_n)\cdots\alpha^{(k-1)n}(a_n) = 0$

$\Rightarrow a_n^3\alpha^{3n}(a_n)\cdots\alpha^{(k-1)n}(a_n)=0$

$\vdots$

$\Rightarrow a_n^k=0$. Therefore $a_n \in ni\ell(R)$. Thus we obtain $[a_0+a_1x+\cdots+a_{n-1}x^{n-1}]^k \in ni\ell(R)[x;\alpha,\delta]$, since R is semicommutative ring and hence $ni\ell(R)$ is an ideal of R, by [8, Lemma 3.1]. So we have $a_{n-1}\alpha^{n-1}(a_{n-1})\cdots\alpha^{(k-1)(n-1)}(a_{n-1}) \in ni\ell(R)$, and similar discussion yields that $a_{n-1}\in ni\ell(R)$. Therefore, by induction, we have $a_i \in ni\ell(R)$ for each $0\le i\le n$.

Conversely, suppose that $a_i^{m_i}=0$, for $i=0,1,\cdots,n$. Let $k=\sum m_i+1$. Thus $[a_0+a_1x+\cdots+a_nx^n]^k=\sum(a_0^{i_{0_1}}(a_1x)^{i_{1_1}}\cdots(a_nx^n)^{i_{n_1}})(a_0^{i_{0_2}}(a_1x)^{i_{1_2}}\cdots(a_nx^n)^{i_{n_2}})\cdots(a_0^{i_{0_k}}(a_1x)^{i_{1_k}}\cdots(a_nx^n)^{i_{n_k}})$, where $\sum\limits_{l=0}^{n} i_{l_r}=1$, for $1\le r\le k$ and $0\le i_{l_r}\le 1$. Each coefficient of $[f(x)]^k$ is a sum of elements $((\alpha^{v_{0_1}}(a_0))^{i_{o_1}}\cdots(\alpha^{v_{n_1}}(a_n))^{i_{n_1}})\cdots\big((\alpha^{v_{0_k}}(a_0))^{i_{o_k}}\cdots(\alpha^{v_{n_k}}(a_n))^{i_{n_k}}\big)$, where $i_{0_r}+i_{1_r}+i_{n_r}=1$. It can be easily checked that there exists $a_s\in\{a_0,a_1,\cdots,a_n\}$ such that $i_{s_1}+i_{s_2}+\cdots+i_{s_k}\ge m_s$. Since $a_s^{m_s}=0$, so $a_s^{i_{s_1}+i_{s_2}+\cdots+i_{s_k}}=0$, and hence by Lemma 14 we have $((\alpha^{v_{s_1}}(a_s))^{i_{s_1}}(\alpha^{v_{s_2}}(a_s))^{i_{s_2}}\cdots(\alpha^{v_{s_k}}(a_s))^{i_{s_k}})=0$. Thus each coefficient of $[f(x)]^k$ is zero, since R is semicommutative. Hence $[f(x)]^k=0$, as desired.

Corollary 17. *Let R be a semicommutative ring. Then $f(x)=\sum\limits_{i=0}^{m} a_ix^i$ is a nilpotent element of $R[x]$ if and only if $a_i\in ni\ell(R)$ for all $0\le i\le n$.*

Proposition 18. *Each α-compatible semicommutative ring is α-weak Armendariz.*

Proof. Note that $f(x)g(x)=\left(\sum\limits_{i=0}^{m}a_ix^i\right)\left(\sum\limits_{j=0}^{n}b_jx^j\right)=\sum\limits_{k=0}^{m+n}\left(\sum\limits_{i+j=k}a_i\alpha^i(b_j)\right)x^k=0$. Then we have the following equations:

$$\sum_{i+j=k}a_i\alpha^i(b_j)=0 \qquad , \qquad k=0,1,\cdots,m+n.$$

We will show that $a_ib_j\in ni\ell(R)$ by induction on $i+j$.

If $i+j=0$, then $0=a_0b_0\in ni\ell(R)$ and so $b_0a_0\in ni\ell(R)$.

Now suppose that k is a positive integer such that $a_ib_j\in ni\ell(R)$ when $i+j<k$. We will show that $a_ib_j\in ni\ell(R)$ when $i+j=k$.

We have the equation

$$a_0b_k+a_1\alpha(b_{k-1})+a_2\alpha^2(b_{k-2})+\cdots+a_k\alpha^k(b_0)=0, \qquad (*)$$

since it is the coefficient of x^k in $f(x)g(x)=0$. Multiplying $(*)$ by b_0 from left, we have

$b_0a_k\alpha^k(b_0)=-\big(b_0a_0b_k+b_0a_1\alpha(b_{k-1})+b_0a_2\alpha^2(b_{k-2})+\cdots+b_0a_{k-1}\alpha^{k-1}(b_1)\big)$.

On the other hand, by induction hypothesis we have $a_ib_0\in ni\ell(R)$ for each $0\le i<k$, and so $b_0a_i\in ni\ell(R)$ for each $0\le i<k$. Hence $b_0a_k\alpha^k(b_0)\in ni\ell(R)$, since $ni\ell(R)$ is an ideal of R. Thus $b_0a_k\alpha^k(b_0)\alpha^k(a_k)=b_0a_k\alpha^k(b_0a_k)\in ni\ell(R)$. Then by Lemma 10, we have $b_0a_k\in ni\ell(R)$, since R is α-compatible, and so

$a_k b_0 \in ni\ell(R)$. Multiplying $(*)$ by b_1 from left, and by a similar way as above we have $a_{k-1}b_1 \in ni\ell(R)$.
Continuing this process yields that $a_i b_j \in ni\ell(R)$ when $i + j = k$. Therefore by induction we have $a_i b_j \in ni\ell(R)$ for each i, j.

Corollary 19. *Each α-rigid ring is α-weak Armendariz.*

Example 20. Consider the ring $R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$, where R is an α-rigid ring. The endomorphism α of R is extended to the endomorphism $\bar{\alpha} : R_3 \to R_3$ defined by $\bar{\alpha}((a_{ij})) = (\alpha(a_{ij}))$. The ring R_3 is not reduced and hence it is not $\bar{\alpha}$-rigid. But since R is α-rigid, R is α-weak Armendariz and hence R_3 is $\bar{\alpha}$-weak Armendariz, by Propositions 4 and 2(2).

Corollary 21. [8, Corollary 3.4]. *Semicommutative rings are weak Armendariz.*

Theorem 22. *Let R be a semicommutative ring and α be an endomorphism of a ring R such that R is α-compatible. If, for some positive integer t, $\alpha^t = id_R$, then $R[x; \alpha]$ is weak Armendariz.*

Proof. Let $F(y) = \sum_{i=0}^{m} f_i y^i$, $G(y) = \sum_{j=0}^{n} g_j y^j \in (R[x; \alpha][y])$ such that $F(y)G(y) = 0$, where $f_i = \sum_{s=0}^{p_i} a_{is} x^s$, $g_j = \sum_{t=0}^{q_j} b_{jt} x^t \in R[x; \alpha]$. Let $k = \sum_i deg(f_i) + \sum_j deg(g_j)$, where the degree is as polynomial in x and the degree of zero polynomial is taken to be 0. Then $F(x^{tk}) = \sum_{i=0}^{m} f_i x^{itk}$, $G(x^{tk}) = \sum_{j=0}^{n} g_j x^{jtk} \in R[x; \alpha]$. Since $F(y)G(y) = 0$ and $\alpha^t = id_R$, so $F(x^k)G(x^k) = 0$. Then by Proposition 18, $a_{is} b_{jt} \in ni\ell(R)$ for all $0 \le i \le m$, $0 \le s \le p_i$, $0 \le j \le n$, $0 \le t \le q_j$. Hence by Lemma 13, $a_{is}\alpha^i(b_{jt}) \in ni\ell(R)$. Since R is semicommutative, $ni\ell(R)$ is an ideal of R and so $\sum_{s+t=k} a_{is}\alpha^i(b_{jt}) \in ni\ell(R)$. Then $f_i g_j \in ni\ell(R[x; \alpha])$, by Proposition 16. Therefore $R[x; \alpha]$ is weak Armendariz ring.

Corollary 23. [8, Theorem 3.8] *If R is a semicommutative ring, then $R[x]$ is weak Armendariz ring.*

Theorem 24. *Let R be a semicommutative and α-compatible ring. If $R[x; \alpha]$ is weak Armendariz, then R is α-weak Armendariz.*

Proof. Suppose that $R[x; \alpha]$ is weak Armendariz ring and $p(x)q(x) = 0$, where $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]$. Then $f(y)g(y) = 0$, for $f(y) = a_0 + (a_1 x)y + \cdots + (a_m x^m)y^m$ and $g(y) = b_0 + (b_1 x)y + \cdots + (b_n x^n)y^n \in (R[x;\alpha])[y]$. Since $R[x;\alpha]$ is weak Armendariz, $a_i x^i b_j x^j \in ni\ell(R[x;\alpha])$, for each i, j. Then by Proposition 16, $a_i \alpha^i(b_j) \in ni\ell(R)$ for each i, j. So by Lemma 13, $a_i b_j \in ni\ell(R)$, for each i, j. Therefore R is α-weak Armendariz ring.

Corollary 25. *If R is a semicommutative ring, then R is weak Armendariz ring if and only if $R[x]$ is weak Armendariz.*

Recall that if α is an endomorphism of a ring R, then the map $\bar{\alpha} : R[x] \to R[x]$ defined by $\bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$, and clearly this map extends α.

Theorem 26. *Let R be a semicommutative and α-compatible ring. Then $R[x]$ is $\bar{\alpha}$-weak Armendariz ring.*

Proof. Let $F(y) = \sum_{i=0}^{p} f_i y^i$, $G(y) = \sum_{j=0}^{q} g_j y^j \in (R[x])[y;\bar{\alpha}]$ such that $F(y)G(y) = 0$, where $f_i = \sum_{s=0}^{m_i} a_{is} x^s$, $g_j = \sum_{t=0}^{n_j} b_{jt} x^t \in R[x]$. Let $m = Max\{m_i \mid 0 \le i \le p\}$ and $n = Max\{n_j \mid 0 \le j \le q\}$. Then we can assume $f_i = \sum_{s=0}^{m} a_{is} x^s$ and $g_j = \sum_{t=0}^{n} b_{jt} x^t \in R[x]$. Hence $F(y) = \sum_{i=0}^{p}(\sum_{s=0}^{m} a_{is} x^s) y^i = \sum_{s=0}^{m}(\sum_{i=0}^{p} a_{is} y^i) x^s$ and also
$G(y) = \sum_{j=0}^{q}(\sum_{t=0}^{n} b_{jt} x^t) y^j = \sum_{t=0}^{n}(\sum_{j=0}^{q} b_{jt} y^j) x^t$. Since $F(y)G(y) = 0$, hence we have

$$\sum_{s+t=k}(\sum_{i=0}^{p} a_{is} y^i)(\sum_{j=0}^{q} b_{jt} y^j) = 0, \text{ for } 0 \le k \le m+n. \qquad (*)$$

We will show by induction on $s+t$ that $a_{is} b_{jt} \in ni\ell(R)$ for any $0 \le i \le p$, any $0 \le j \le q$, and any s, t with $s+t = 0, 1, \cdots, m+n$.
If $s+t = 0$, then $s = t = 0$. Thus $(\sum_{i=0}^{p} a_{i0} y^i)(\sum_{j=0}^{q} b_{j0} y^j) = 0$. Since R is semicommutative and α-compatible, so by Proposition 18, R is α- weak Armendariz. Thus $a_{i0} b_{j0} \in ni\ell(R)$ for any $0 \le i \le p$, any $0 \le j \le q$.
Now suppose that $k \le m+n$ is such that $a_{is} b_{jt} \in ni\ell(R)$ for any $0 \le i \le p$, any $0 \le j \le q$, and any s, t with $s+t < k$. We will show that $a_{is} b_{jt} \in ni\ell(R)$ for any $0 \le i \le p$, any $0 \le j \le q$, and any s, t with $s+t = k$. From $(*)$ we have

$$0 = \sum_{s+t=k}(\sum_{i=0}^{p} a_{is} y^i)(\sum_{j=0}^{q} b_{jt} y^j) = \sum_{s+t=k}\sum_{l=0}^{p+q}(\sum_{i+j=l} a_{is} \alpha^i(b_{jt})) y^l$$

$$= \sum_{l=0}^{p+q} (\sum_{s+t=k} (\sum_{i+j=l} a_{is} \alpha^i (b_{jt})) y^l = \sum_{l=0}^{p+q} (\sum_{i+j=l} (\sum_{s+t=k} a_{is} \alpha^i (b_{jt})) y^l.$$

Thus

$$\sum_{s+t=k} a_{0s} b_{0t} = 0,$$

$$\sum_{s+t=k} a_{0s} b_{1t} + \sum_{s+t=k} a_{1s} \alpha(b_{0t}) = 0,$$

$$\dots\dots\dots\dots$$

$$\sum_{s+t=k} a_{0s} b_{lt} + \sum_{s+t=k} a_{1s} \alpha(b_{(l-1)t}) + \cdots + \sum_{s+t=k} a_{ls} \alpha^l (b_{0t}) = 0,$$

$$\sum_{s+t=k} a_{ps} \alpha^p (b_{qt}) = 0.$$

If $s < k$, then by induction hypothesis, $a_{0s} b_{00} \in ni\ell(R)$ and so $b_{00} a_{0s} \in ni\ell(R)$. Hence $b_{00} a_{00} b_{0k} + b_{00} a_{01} b_{0k-1} + \cdots + b_{00} a_{0k-1} b_{01} \in ni\ell(R)$, since R is semicommutative. If we multiply $\sum_{s+t=k} a_{0s} b_{0t} = 0$ on left side by b_{00}, then it follows that $b_{00} a_{0k} b_{00} \in ni\ell(R)$, and so $b_{00} a_{0k} \in ni\ell(R)$ and $a_{0k} b_{00} \in ni\ell(R)$. If we multiply $\sum_{s+t=k} a_{0s} b_{0t} = 0$ on the left side by b_{01}, then we have $b_{01} a_{0k-1} b_{01} = -(b_{01} a_{00} b_{0k} + b_{01} a_{01} b_{0k-1} + \cdots + b_{01} a_{0k-2} b_{02}) - b_{01} a_{0k} b_{00} = -(b_{01} a_{00}) b_{0k} - (b_{01} a_{01}) b_{0k-1} - \cdots - (b_{01} a_{0k-2}) b_{02} - b_{01}(a_{0k} b_{00}) \in ni\ell(R)$, since R is semicommutative. Thus $a_{0k-1} b_{01} \in ni\ell(R)$. Similarly, we can show that $a_{0k-2} b_{02} \in ni\ell(R), \cdots, a_{00} b_{0k} \in ni\ell(R)$. So we show that $a_{is} b_{jt} \in ni\ell(R)$ for any s,t with $s + t = k$ and any i,j with $i + j = 0$. Suppose that $l \leq p + q$ is such that $a_{is} b_{jt} \in ni\ell(R)$ for any s, t with $s + t = k$ and any i, j with $i + j < l$. We will show that $a_{is} b_{jt} \in ni\ell(R)$ for any s, t with $s + t = k$ and any i, j with $i + j = l$. If $s < k$, then by induction hypothesis, $a_{is} b_{00} \in ni\ell(R)$. So $b_{00} a_{is} \in ni\ell(R)$. If $i < l$, then by induction hypothesis on l, $a_{ik} b_{00} \in ni\ell(R)$ for any $i < l$, and so $b_{00} a_{ik} \in ni\ell(R)$ for any $i < l$. Multiplying $\sum_{s+t=k} a_{0s} b_{lt} + \sum_{s+t=k} a_{1s} \alpha(b_{l-1t}) + \cdots + \sum_{s+t=k} a_{ls} \alpha^l (b_{0t}) = 0$ on left side by b_{00}, we have $b_{00} a_{lk} \alpha^l (b_{00}) \in ni\ell(R)$, since $ni\ell(R)$ is an ideal of R. Thus $b_{00} a_{lk} \alpha^l (b_{00}) \alpha^l (a_{lk}) = b_{00} a_{lk} \alpha^l (b_{00} a_{lk}) \in ni\ell(R)$. Thus $b_{00} a_{lk} \in ni\ell(R)$ by Lemma 13, and so $a_{lk} b_{00} \in ni\ell(R)$. Similarly, we can show that $a_{is} b_{jt} \in ni\ell(R)$ for any s, t with $s + t = k$ and any i, j with $i + j = l$. Therefore, by induction, we have $a_{is} b_{jt} \in ni\ell(R)$ for any $0 \leq i \leq p$, and $0 \leq j \leq q$ and any s, t with $s + t = 0, 1, \cdots, m + n$. Hence $\sum_{s+t=k} a_{is} b_{jt} \in ni\ell(R)$, since R is semicommutative. Thus $f_i g_j \in ni\ell(R[x])$, by Proposition 16. Therefore $R[x]$ is $\bar{\alpha}$-weak Armendariz.

ACKNOWLEDGEMENTS. The authors would like to thank the referee for helpful comments and suggestions.

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Received: August, 2010