Weak Armendariz Skew Polynomial Rings

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Abstract

A ring $R$ is called weak Armendariz if whenever the product of any two polynomials in $R[x]$ over $R$ is zero, then the product of any pair of coefficients from the two polynomials be a nilpotent element of $R$. For a ring endomorphism $\alpha$, we introduce the notion of $\alpha$-weak Armendariz rings by considering the polynomials in the skew polynomial ring $R[x; \alpha]$ instead of $R[x]$, which are a generalization of weak Armendariz rings and $\alpha$-Armendariz rings. Basic properties of $\alpha$-weak Armendariz rings are observed, and connections of properties of an $\alpha$-weak Armendariz ring $R$ with those of $R[x; \alpha]$ are investigated. As a consequence our results improve not only some results of Liu [8], but also some known results on Armendariz rings.

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Throughout this paper $R$ denotes an associative ring with unity. For a ring $R$, we denote by $\text{nil}(R)$ the set of all nilpotent elements of $R$ and by $R[x]$ the polynomial ring with an indeterminate $x$ over $R$.

By [11], a ring $R$ is called an Armendariz ring if whenever $f(x)g(x) = 0$ where $f(x) = a_0 + a_1 x + \cdots + a_m x^m$, $g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$, then $a_i b_j = 0$ for each $i, j$. The name Armendariz ring is chosen because Armendariz
[2] had noted that a reduced ring (i.e., a ring without nonzero nilpotent elements) satisfies this condition. Recall that a ring $R$ is called reversible if $ab = 0$ implies $ba = 0$, for all $a, b \in R$; $R$ is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. Reduced rings are clearly reversible and reversible rings are semicommutative, but the converse is not true in general[10].

Liu and Zhao [8] have studied a generalization of Armendariz rings, which they called weak Armendariz rings. A ring $R$ is called a weak-Armendariz ring if for each $a, b \in R$, if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j \in \text{nil}(R)$ for each $i, j$. Each semicommutative ring is weak Armendariz by [8] and so weak Armendariz rings are a common generalization of semicommutative rings and Armendariz rings.

The Armendariz property of a ring was extended to one of skew polynomials [4,5]. For an endomorphism $\alpha$ of a ring $R$, the skew polynomial ring $R[x; \alpha]$ consists of the polynomials in $x$ with coefficients in $R$ written on the left, subject to the relation $x r = \alpha(r)x$ for all $r \in R$. A ring $R$ is called $\alpha$-Armendariz (resp., $\alpha$-skew Armendariz) if for $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ (resp., $a_i \alpha^j(b_j) = 0$) for all $0 \leq i \leq m$ and $0 \leq j \leq n$ [5, Definition 1.1](resp., [4, Definition]). According to Hashemi and Moussavi [3], a ring $R$ is said to be $\alpha$-compatible if for each $a, b \in R$, $ab = 0 \iff a \alpha(b) = 0$. For $\alpha$-compatible rings the above mentioned concepts are equivalent. According to Krempa [6], an endomorphism $\alpha$ of a ring $R$ is called to be rigid if $a \alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is called $\alpha$-rigid if there exist a rigid endomorphism $\alpha$ of $R$. By [3, Lemma 2.2], a ring $R$ is $\alpha$-rigid if and only if $R$ is $\alpha$-compatible and reduced. Also by [4, Proposition 3], if $R$ is $\alpha$-rigid ring, then $R[x; \alpha]$ is reduced.

We are motivated to introduce the notion of $\alpha$-weak Armendariz ring $R$ with respect to an endomorphism $\alpha$ of $R$. This notion extends both ring weak Armendariz rings and $\alpha$-Armendariz rings. We do this by considering the weak Armendariz condition on polynomials in $R[x; \alpha]$ instead of $R[x]$. This provides us with an opportunity to study weak Armendariz rings in a general setting, and several known results on weak Armendariz rings are obtained as corollaries. We start with the following definition.

**Definition 1.** Let $\alpha$ be an endomorphism of a ring $R$. The ring $R$ is called $\alpha$-weak Armendariz (resp., $\alpha$-skew weak Armendariz) if for $f(x) = a_0 + a_1 x + \cdots + a_m x^m, g(x) = b_0 + b_1 x + \cdots + b_n x^n \in R[x; \alpha]$ satisfy $f(x)g(x) = 0$, then $a_i b_j \in \text{nil}(R)$ (resp., $a_i \alpha^j(b_j) \in \text{nil}(R)$) for all $0 \leq i \leq m$ and $0 \leq j \leq n$.

It is clear that a ring $R$ is weak Armendariz if $R$ is $id_R$-weak Armendariz, where $id_R$ is the identity endomorphism of $R$. 

[\alpha]
Proposition 2. For an endomorphism $\alpha$ of a ring $R$, we have the following statements:

1. If $R$ is $\alpha$-Armendariz (resp., $\alpha$-skew Armendariz) ring, then $R$ is $\alpha$-weak Armendariz (resp., $\alpha$-skew weak Armendariz).

2. Every subring $S$ with $\alpha(S) \subseteq S$ of an $\alpha$-weak Armendariz ring is also $\alpha$-weak Armendariz.

Proof. It is clear by definition.

The following example shows that there exists an endomorphism $\alpha$ of a weak Armendariz ring $R$ such that $R$ is not $\alpha$-weak Armendariz.

Example 3. Let $R = R_1 \oplus R_2$, where $R_1$ and $R_2$ be any reduced rings. Then $R$ is a semicommutative and so $R$ is weak Armendariz. Let $\alpha : R \to R$ be an endomorphism defined by $\alpha((a, b)) = (b, a)$. Let $f(x) = (0, 1) - (0, 1)x$, $\overline{g}(x) = (1, 0) + (0, 1)x \in R[x; \alpha]$. Then $f(x)g(x) = 0$, but $(0, 1)(0, 1) = (0, 1) \notin \text{nil}(R)$. Therefore $R$ is neither $\alpha$-weak Armendariz nor $\alpha$-skew weak Armendariz.

Any endomorphism $\alpha$ of $R$ can be extended to an endomorphism $\overline{\alpha}$ of $T_n(R)$ defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$.

Proposition 4. Let $\alpha$ be an endomorphism of a ring $R$. Then $R$ is $\alpha$-weak Armendariz if and only if, for any $n$, $T_n(R)$ is $\alpha$-weak Armendariz.

Proof. We only prove necessity, since subrings of $\alpha$-weak Armendariz rings is also $\alpha$-weak Armendariz. Note that $T_n(R)[x; \overline{\alpha}] \cong T_n(R[x; \alpha])$. Let $f(x) = \sum_{i=0}^{p} A_i x^i$ and $g(x) = \sum_{j=0}^{q} B_j x^j \in T_n(R[x; \overline{\alpha}])$ are such that $f(x)g(x) = 0$. Let

$$A_i = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ 0 & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^i \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b_{22}^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn}^j \end{pmatrix},$$

for $0 \leq i \leq p$, $0 \leq j \leq q$. Then $f_s(x) = \sum_{i=0}^{p} a_{ss}^i x^i$ and $g_s(x) = \sum_{j=0}^{q} b_{ss}^j x^j \in R[x; \alpha]$ and $f_s(x)g_s(x) = 0$, for each $1 \leq s \leq n$. Since $R$ is $\alpha$-weak Armendariz ring, there exists $m_{ij} \in \mathbb{N}$ such that $(a_{ss}^i b_{ss}^j)^{m_{ij}} = 0$, for any $s$ and any $i, j$. Let $m_{ij} = \max\{m_{ij} | 1 \leq s \leq n\}$. Then $[(A_i B_j)^{m_{ij}}]^n = 0$. Therefore $T_n(R)$ is $\overline{\alpha}$-weak Armendariz.

Corollary 5. [8, Proposition 2.2] A ring $R$ is a weak Armendariz ring if and only if, for any $n$, $T_n(R)$ is weak Armendariz.
Definition 6. For a ring $R$, consider the following set of triangular matrices:

$$T(R, n) := \left\{ \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} \mid a_i \in R \right\}$$

with $n \geq 2$.

It is easy to see that $T(R, n)$ is a subring of the triangular matrix rings, with matrix addition and multiplication. We can denote elements of $T(R, n)$ by $(a_0, a_1, \ldots, a_{n-1})$, then $T(R, n)$ is a ring with addition point-wise and multiplication given by

$$(a_0, a_1, \ldots, a_{n-1})(b_0, b_1, \ldots, b_{n-1}) = (a_0b_0, a_0b_1 + a_1b_0, \ldots, a_0b_{n-1} + \cdots + a_{n-1}b_0),$$

for each $a_i, b_j \in R$.

On the other hand, there is a ring isomorphism $\varphi : R[x]/\langle x^n \rangle \to T(R, n)$, given by $\varphi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, $0 \leq i \leq n - 1$. So $T(R, n) \cong R[x]/\langle x^n \rangle$, where $R[x]$ is the ring of polynomial in an indeterminate $x$ and $\langle x^n \rangle$ is the ideal generated by $x$.

Theorem 7. A ring $R$ is $\alpha$-weak Armendariz ring if and only if, for any $n$, $\frac{R[x]}{\langle x^n \rangle}$ is $\overline{\alpha}$-weak Armendariz.

Proof. We only prove necessity, since subrings of $\alpha$-weak Armendariz rings is also $\alpha$-weak Armendariz. Note that $nil(T(R, n)) = (nil(R), R, \ldots, R)$ and also $T(R, n)[x; \overline{\alpha}] \cong T(R[x; \alpha], n)$. Let $F(x) = \sum_{i=0}^{m} A_i x_i$, $G(x) = \sum_{j=0}^{n} B_j x_j \in T(R, n)[x; \overline{\alpha}]$ are such that $F(x)G(x) = 0$. Let

$$A_i = \begin{pmatrix} a_{i0} & a_{i1} & \cdots & a_{i(n-2)} & a_{i(n-1)} \\ 0 & a_{i0} & a_{i1} & \cdots & a_{i(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_{i0} & \cdots & a_{i1} \\ 0 & 0 & \cdots & \cdots & a_{i0} \end{pmatrix}, B_j = \begin{pmatrix} b_{j0} & b_{j1} & \cdots & b_{j(n-2)} & b_{j(n-1)} \\ 0 & b_{j0} & b_{j1} & \cdots & b_{j(n-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & b_{j0} & \cdots & b_{j1} \\ 0 & 0 & \cdots & 0 & b_{j0} \end{pmatrix}$$

for $0 \leq i \leq m$, $0 \leq j \leq n$. Then $f_0(x) = \sum_{i=0}^{m} a_{i0}x_i$ and $g_0(x) = \sum_{j=0}^{n} b_{j0}x_j \in R[x; \alpha]$ and $f_0(x)g_0(x) = 0$. Since $R$ is $\alpha$-weak Armendariz ring, there exists $m_{ij} \in \mathbb{N}$ such that $(a_{i0}b_{j0})^{m_{ij}} = 0$, for any $i, j$. Then $A_iB_j \in nil(T(R, n))$. Therefore $\frac{R[x]}{\langle x^n \rangle}$ is $\overline{\alpha}$-weak Armendariz.

If we take $\alpha = id_R$, we deduce:

Corollary 8. A ring $R$ is weak Armendariz if and only if, for any $n$, $\frac{R[x]}{\langle x^n \rangle}$ is weak Armendariz.

Corollary 9. [8, Theorem 3.9] If $R$ is a semicommutative ring, then for any
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Let \( R[\frac{R[x]}{(x^n)}] \) is weak Armendariz.

Recall that for a ring \( R \) and an \((R,R)\)-bimodule \( M \), the trivial extension of \( R \) by \( M \) is the ring \( T(R,M) = R \oplus M \) with the usual addition and the multiplication \((r_1,m_1)(r_2,m_2) = (r_1r_2,r_1m_2 + r_2m_1)\). This is isomorphic to the ring of all matrices \( \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \), where \( r \in R \) and \( m \in M \) and the usual matrix operations are used.

**Corollary 10.** Let \( \alpha \) be an endomorphism of a ring \( R \). Then \( R \) is \( \alpha \)-weak Armendariz if and only if, \( T(R,R) \) is \( \bar{\alpha} \)-weak Armendariz.

**Proof.** Observe that \( R[\frac{R[x]}{(x^n)}] \cong T(R,R) \).

**Example 11.** Consider the ring \( R_4 = \left\{ \begin{pmatrix} a_{12} & a_{13} & a_{14} \\ 0 & a_{23} & a_{24} \\ 0 & 0 & a_{34} \\ 0 & 0 & 0 \end{pmatrix} \mid a_{ij} \in R \right\} \), where \( R \) is an \( \alpha \)-rigid ring. The endomorphism \( \alpha \) of \( R \) is extended to the endomorphism \( \bar{\alpha} : R_4 \rightarrow R_4 \) defined by \( \bar{\alpha}((a_{ij})) = (\alpha(a_{ij})) \). The ring \( R_4 \) is not \( \bar{\alpha} \)-Armendariz by [4, Example 18] and [5, Theorem 1.8]. But since \( R \) is \( \alpha \)-rigid, \( R \) is \( \alpha \)-weak Armendariz and hence \( R_4 \) is \( \bar{\alpha} \)-weak Armendariz by Propositions 4 and 2(2).

**Lemma 12.** [3, Lemma 3.2] Let \( R \) be an \( \alpha \)-compatible ring. Then we have the following:

(i) If \( ab = 0 \), then \( a\alpha^n(b) = \alpha^n(a)b = 0 \), for all positive integers \( n \).

(ii) If \( \alpha^k(a)b = 0 \) for some positive integer \( k \), then \( ab = 0 \).

We now state another lemma that will be helpful to us in the sequel.

**Lemma 13.** Let \( R \) be an \( \alpha \)-compatible ring. Then we have the following:

(i) If \( ab \in \text{nil}(R) \), then \( a\alpha^m(b) \in \text{nil}(R) \) and \( \alpha^n(a)b \in \text{nil}(R) \), for all positive integers \( m, n \).

(ii) If \( \alpha^k(a)b \in \text{nil}(R) \) or \( a\alpha^l(b) \in \text{nil}(R) \) for some positive integers \( k, l \), then \( ab \in \text{nil}(R) \).

**Proof.** Since \( ab \in \text{nil}(R) \), there exist some positive integer \( k \) such that \( (ab)^k = abab \cdots ab = 0 \). Using \( \alpha \)-compatibility of \( R \), we have:

\[
ababab \cdots abab = 0 \\
\Leftrightarrow ababab \cdots abaa^m(b) = 0 \\
\Leftrightarrow ababab \cdots a\alpha^m(baa^m(b)) = 0
\]
\[ \Leftrightarrow ababab \cdots a\alpha^m(b)\alpha^m(a\alpha^m(b)) = 0 \]
\[ \Leftrightarrow ababab \cdots a\alpha^m(b)a\alpha^m(b) = 0 \]
\[ \vdots \]
\[ \Leftrightarrow [a\alpha^m(b)]^k = 0. \] Therefore \(a\alpha^m(b)\in\text{nil}(R)\). By a similar way, one can prove the other cases.

**Lemma 14.** Let \(R\) be \(\alpha\)-compatible ring. Then \(a^{k_1+k_2+\cdots+k_n} = 0\), implies that 
\[ (\alpha^{v_1}(a))^{k_1}(\alpha^{v_2}(a))^{k_2} \cdots (\alpha^{v_n}(a))^{k_n} = 0, \] where \(v_p\) (\(1 \leq p \leq n\)) is a nonnegative integer.

**Proof.** Using Lemma 12 and \(\alpha\)-compatibility of \(R\), we have:
\[ a^{k_1+k_2+\cdots+k_n} = 0 \]
\[ \Rightarrow a^{k_1}a^{k_2} \cdots a^{k_n} = 0 \]
\[ \Rightarrow a^{k_1}a^{k_2} \cdots a^{k_{n-1}}\alpha^{v_n}(a^{k_n}) = 0 \]
\[ \Rightarrow a^{k_1}a^{k_2} \cdots a^{k_{n-1}}(\alpha^{v_n}(a))^{k_n} = 0 \]
\[ \Rightarrow \alpha^{v_n-1}(a^{k_1}a^{k_2} \cdots a^{k_{n-2}})(\alpha^{v_n-1}(a^{k_{n-1}})(\alpha^{v_n}(a))^{k_n} = 0 \]
\[ \Rightarrow \alpha^{v_n-1}(a^{k_1}a^{k_2} \cdots a^{k_{n-2}})(\alpha^{v_n-1}(a^{k_{n-1}})(\alpha^{v_n}(a))^{k_n} = 0 \]
\[ \Rightarrow a^{k_1}a^{k_2} \cdots a^{k_{n-2}}(\alpha^{v_n-1}(a))^{k_{n-1}}(\alpha^{v_n}(a))^{k_n} = 0 \]
\[ \vdots \]
\[ \Rightarrow (\alpha^{v_1}(a))^{k_1}(\alpha^{v_2}(a))^{k_2} \cdots (\alpha^{v_n}(a))^{k_n} = 0. \]

Recall that an ideal \(I\) of a ring \(R\) is called \(\alpha\)-invariant if \(\alpha(I) \subseteq I\). If an ideal \(I\) of \(R\) is \(\alpha\)-invariant, then \(I[x;\alpha]\) is an ideal of \(R[x;\alpha]\), as for any \(a \in I\), \(\alpha^j(a) \in I\) for all positive integers \(j\).

**Corollary 15.** Let \(R\) be \(\alpha\)-compatible ring. Then \(\text{nil}(R)[x;\alpha]\) is an ideal of \(R[x;\alpha]\).

**Proposition 16.** Let \(R\) be semicommutative and \(\alpha\)-compatible ring and \(f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha]\). Then \(f(x) \in \text{nil}(R[x;\alpha])\) if and only if \(a_i \in \text{nil}(R)\) for all \(0 \leq i \leq n\).

**Proof.** Suppose \(f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha]\) and there exist some positive integer \(k\) such that \([f(x)]^k = 0\). Then \(a_n\alpha^n(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0\), since it is the leading coefficient of \([f(x)]^k\). Using Lemma 11 and \(\alpha\)-compatibility of \(R\), we have:
\[ a_n\alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \]
\[ \Rightarrow \alpha^n(a_n)\alpha^n(a_n)\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \]
\[ \Rightarrow \alpha^n(a_n^2)\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \]
\[ \Rightarrow a_n^2\alpha^{2n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \]
\[ \Rightarrow \alpha^{2n}(a_n^3)\alpha^{3n}(a_n) \cdots \alpha^{(k-1)n}(a_n) = 0 \]
Each coefficient of $a^3 \alpha^3 (a_n) \cdots \alpha^{(k-1)n}(a_n) = 0$

\[ \Rightarrow a_n^k = 0. \] Therefore, $a_n \in \text{nil}(R)$. Thus we obtain $[a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}]^k \in \text{nil}(R)[x; \alpha, \delta]$, since $R$ is semicommutative ring and hence $\text{nil}(R)$ is an ideal of $R$, by [8, Lemma 3.1]. So we have $a_{n-1}^n \alpha^{(n-1)(n-1)}(a_{n-1}) \in \text{nil}(R)$, and similar discussion yields that $a_{n-1} \in \text{nil}(R)$. Therefore, by induction, we have $a_i \in \text{nil}(R)$ for each $0 \leq i \leq n$.

Conversely, suppose that $\alpha^{n_i} = 0$, for $i = 0, 1, \ldots, n$. Let $k = \sum m_i + 1$. Thus
\[ [a_0 + a_1 x + \cdots + a_n x^n]^k = \sum (a_0^{i_0} (a_1 x)^{i_1} \cdots (a_n x^n)^{i_n}) (a_0^{i_0} (a_1 x)^{i_2} \cdots (a_n x^n)^{i_2}) \cdots (a_0^{i_0} (a_1 x)^{i_k} \cdots (a_n x^n)^{i_k}), \]
where $\sum i_n = 1$, for $1 \leq k \leq n$ and $0 \leq i_n \leq 1$.

Each coefficient of $[f(x)]^k$ is a sum of elements $((\alpha^{v_0} (a_0))^{i_0} \cdots (\alpha^{v_1} (a_n))^{i_n}) \cdots (\alpha^{v_k} (a_s))^{i_k}$, where $i_0 + i_1 + \cdots + i_k = 1$. It can be easily checked that there exists $a_s \in \{a_0, a_1, \cdots, a_n\}$ such that $i_1 + i_2 + \cdots + i_k \geq m_s$.

Since $\alpha^{m_s} = 0$, so $a_s^{i_1 + i_2 + \cdots + i_k} = 0$, and hence by Lemma 14 we have $((\alpha^{v_1} (a_s))^{i_0} (\alpha^{v_2} (a_s))^{i_2} \cdots (\alpha^{v_k} (a_s))^{i_k}) = 0$. Thus each coefficient of $[f(x)]^k$ is zero, since $R$ is semicommutative. Hence $[f(x)]^k = 0$, as desired.

**Corollary 17.** Let $R$ be a semicommutative ring. Then $f(x) = \sum a_i x^i$ is a nilpotent element of $R[x]$ if and only if $a_i \in \text{nil}(R)$ for all $0 \leq i \leq n$.

**Proposition 18.** Each $\alpha$-compatible semicommutative ring is $\alpha$-weak Armendariz.

**Proof.** Note that $f(x)g(x) = \left( \sum_{i=0}^{m} a_i x^i \right) \left( \sum_{j=0}^{n} b_j x^j \right) = \sum_{k=0}^{m+n} \left( \sum_{i+j=k} a_i \alpha^i(b_j) \right) x^k = 0$. Then we have the following equations:

\[ \sum_{i+j=k} a_i \alpha^i(b_j) = 0, \quad k = 0, 1, \ldots, m+n. \]

We will show that $a_i b_j \in \text{nil}(R)$ by induction on $i+j$.

If $i+j = 0$, then $0 = a_0 b_0 \in \text{nil}(R)$ and so $a_0 b_0 \in \text{nil}(R)$.

Now suppose that $k$ is a positive integer such that $a_i b_j \in \text{nil}(R)$ when $i+j < k$.

We will show that $a_i b_j \in \text{nil}(R)$ when $i+j = k$.

We have the equation

\[ a_0 b_k + a_1 \alpha(b_{k-1}) + a_2 \alpha^2(b_{k-2}) + \cdots + a_k \alpha^k(b_0) = 0, \quad (*) \]

since it is the coefficient of $x^k$ in $f(x)g(x) = 0$. Multiplying (*) by $b_0$ from left, we have

\[ b_0 a_k \alpha^k(b_0) = - (b_0 a_0 b_k + b_0 a_1 \alpha(b_{k-1}) + b_0 a_2 \alpha^2(b_{k-2}) + \cdots + b_0 a_{k-1} \alpha^{k-1}(b_1)). \]

On the other hand, by induction hypothesis we have $a_i b_i \in \text{nil}(R)$ for each $0 \leq i < k$, and so $b_0 a_i \in \text{nil}(R)$ for each $0 \leq i < k$. Hence $b_0 a_k \alpha^k(b_0) \in \text{nil}(R)$, since $\text{nil}(R)$ is an ideal of $R$. Thus $b_0 a_k \alpha^k(b_0) = b_0 a_k \alpha^k(b_0 a_k) \in \text{nil}(R)$.

Then by Lemma 10, we have $b_0 a_k \in \text{nil}(R)$, since $R$ is $\alpha$-compatible, and so
$a_kb_0 \in \text{nil}(R)$. Multiplying (*) by $b_1$ from left, and by a similar way as above we have $a_{k-1}b_1 \in \text{nil}(R)$.

Continuing this process yields that $a_ib_j \in \text{nil}(R)$ when $i + j = k$. Therefore by induction we have $a_ib_j \in \text{nil}(R)$ for each $i, j$.

**Corollary 19.** Each $\alpha$-rigid ring is $\alpha$-weak Armendariz.

**Example 20.** Consider the ring $R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$, where $R$ is an $\alpha$-rigid ring. The endomorphism $\alpha$ of $R$ is extended to the endomorphism $\overline{\alpha} : R_3 \to R_3$ defined by $\overline{\alpha}(\begin{pmatrix} a_{ij} \end{pmatrix}) = (\alpha(a_{ij}))$. The ring $R_3$ is not reduced and hence it is not $\overline{\alpha}$-rigid. But since $R$ is $\alpha$-rigid, $R$ is $\alpha$-weak Armendariz and hence $R_3$ is $\overline{\alpha}$-weak Armendariz, by Propositions 4 and 2(2).

**Corollary 21.** [8, Corollary 3.4]. Semicommutative rings are weak Armendariz.

**Theorem 22.** Let $R$ be a semicommutative ring and $\alpha$ be an endomorphism of a ring $R$ such that $R$ is $\alpha$-compatible. If, for some positive integer $t$, $\alpha^t = id_R$, then $R[x; \alpha]$ is weak Armendariz.

**Proof.** Let $F(y) = \sum_{i=0}^{m} f_i y^i, G(y) = \sum_{j=0}^{n} g_j y^j \in (R[x; \alpha][y])$ such that $F(y)G(y) = 0$, where $f_i = \sum_{s=0}^{p_i} a_{is} x^s, g_j = \sum_{t=0}^{q_j} b_{jt} x^t \in R[x; \alpha]$. Let $k = \sum deg(f_i) + \sum deg(g_j)$, where the degree is as polynomial in $x$ and the degree of zero polynomial is taken to be 0. Then $F(x^k) = \sum_{i=0}^{m} f_i x^{itk}, G(x^k) = \sum_{j=0}^{n} g_j x^{jtk} \in R[x; \alpha]$.

Since $F(y)G(y) = 0$ and $\alpha^t = id_R$, so $F(x^k)G(x^k) = 0$. Then by Proposition 18, $a_{is}b_{jt} \in \text{nil}(R)$ for all $0 \leq i \leq m, 0 \leq s \leq p_i, 0 \leq j \leq n, 0 \leq t \leq q_j$. Hence by Lemma 13, $a_{is}\alpha^t(b_{jt}) \in \text{nil}(R)$. Since $R$ is semicommutative, $\text{nil}(R)$ is an ideal of $R$ and so $\sum_{s+t=k} a_{is}\alpha^t(b_{jt}) \in \text{nil}(R)$. Then $f_i g_j \in \text{nil}(R[x; \alpha])$, by Proposition 16. Therefore $R[x; \alpha]$ is weak Armendariz ring.

**Corollary 23.** [8, Theorem 3.8] If $R$ is a semicommutative ring, then $R[x]$ is weak Armendariz ring.

**Theorem 24.** Let $R$ be a semicommutative and $\alpha$-compatible ring. If $R[x; \alpha]$ is weak Armendariz, then $R$ is $\alpha$-weak Armendariz.

**Proof.** Suppose that $R[x; \alpha]$ is weak Armendariz ring and $p(x)q(x) = 0$,
where \( p(x) = \sum_{i=0}^{m} a_i x^i \) and \( q(x) = \sum_{j=0}^{n} b_j x^j \) ∈ \( R[x; \alpha] \). Then \( f(y)g(y) = 0 \), for \( f(y) = a_0 + (a_1 x)y + \cdots + (a_m x^m)y^m \) and \( g(y) = b_0 + (b_1 x)y + \cdots + (b_n x^n)y^n \) ∈ \( (R[x; \alpha])[y] \). Since \( R[x; \alpha] \) is weak Armendariz, \( a_i x^i b_j x^j \in \text{nil}(R[x; \alpha]) \), for each \( i, j \). Then by Proposition 16, \( a_i \alpha^i(b_j) \in \text{nil}(R) \) for each \( i, j \). So by Lemma 13, \( a_i b_j \in \text{nil}(R) \), for each \( i, j \). Therefore \( R \) is \( \alpha \)-weak Armendariz ring.

**Corollary 25.** If \( R \) is a semicommutative ring, then \( R \) is weak Armendariz ring if and only if \( R[x] \) is weak Armendariz.

Recall that if \( \alpha \) is an endomorphism of a ring \( R \), then the map \( \bar{\alpha} : R[x] \to R[x] \) defined by \( \bar{\alpha}(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} \alpha(a_i) x^i \) is an endomorphism of the polynomial ring \( R[x] \), and clearly this map extends \( \alpha \).

**Theorem 26.** Let \( R \) be a semicommutative and \( \alpha \)-compatible ring. Then \( R[x] \) is \( \bar{\alpha} \)-weak Armendariz ring.

**Proof.** Let \( F(y) = \sum_{i=0}^{p} f_i y^i \), \( G(y) = \sum_{j=0}^{q} g_j y^j \) ∈ \( (R[x])[y; \bar{\alpha}] \) such that \( F(y)G(y) = 0 \), where \( f_i = \sum_{s=0}^{m_i} a_{is} x^s \), \( g_j = \sum_{t=0}^{n_j} b_{jt} x^t \) ∈ \( R[x] \). Let \( m = \text{Max}\{m_i \mid 0 \leq i \leq p\} \) and \( n = \text{Max}\{n_j \mid 0 \leq j \leq q\} \). Then we can assume \( f_i = \sum_{s=0}^{m_i} a_{is} x^s \) and \( g_j = \sum_{t=0}^{n_j} b_{jt} x^t \) ∈ \( R[x] \). Hence \( F(y) = \sum_{i=0}^{p} (\sum_{s=0}^{m_i} a_{is} x^s) y^i = \sum_{s=0}^{m} (\sum_{i=0}^{p} a_{is} y^i) x^s \) and also \( G(y) = \sum_{j=0}^{q} (\sum_{t=0}^{n_j} b_{jt} x^t) y^j = \sum_{t=0}^{n} (\sum_{j=0}^{q} b_{jt} y^j) x^t \). Since \( F(y)G(y) = 0 \), hence we have

\[
\sum_{s+t=k} (\sum_{i=0}^{p} a_{is} y^i)(\sum_{j=0}^{q} b_{jt} y^j) = 0, \quad 0 \leq k \leq m+n. \quad (\ast)
\]

We will show by induction on \( s+t \) that \( a_{is} b_{jt} \in \text{nil}(R) \) for any \( 0 \leq i \leq p \), any \( 0 \leq j \leq q \), and any \( s, t \) with \( s+t = 0, 1, \ldots, m+n \).

If \( s+t = 0 \), then \( s = t = 0 \). Thus \( (\sum_{i=0}^{p} a_{i0} y^i)(\sum_{j=0}^{q} b_{j0} y^j) = 0 \). Since \( R \) is semicommutative and \( \alpha \)-compatible, so by Proposition 18, \( R \) is \( \alpha \)- weak Armendariz. Thus \( a_{i0} b_{j0} \in \text{nil}(R) \) for any \( 0 \leq i \leq p \), any \( 0 \leq j \leq q \).

Now suppose that \( k \leq m+n \) is such that \( a_{is} b_{jt} \in \text{nil}(R) \) for any \( 0 \leq i \leq p \), any \( 0 \leq j \leq q \), and any \( s, t \) with \( s+t < k \). We will show that \( a_{is} b_{jt} \in \text{nil}(R) \) for any \( 0 \leq i \leq p \), any \( 0 \leq j \leq q \), and any \( s, t \) with \( s+t = k \). From (\ast) we have

\[
0 = \sum_{s+t=k} (\sum_{i=0}^{p} a_{is} y^i)(\sum_{j=0}^{q} b_{jt} y^j) = \sum_{s+t=k} \sum_{l=0}^{p+q} (\sum_{i+j=l} a_{is} \alpha^i(b_{jt})) y^l
\]
Thus
\[ \sum_{s+t=k} a_{0s}b_{0t} = 0, \]
\[ \sum_{s+t=k} a_{0s}b_{1t} + \sum_{s+t=k} a_{1s}\alpha(b_{0t}) = 0, \]
\[ \ldots \]
\[ \sum_{s+t=k} a_{0s}b_{lt} + \sum_{s+t=k} a_{1s}\alpha(b_{(l-1)t}) + \cdots + \sum_{s+t=k} a_{ls}\alpha(b_{0t}) = 0, \]
\[ \sum_{s+t=k} a_{ps}\alpha^p(b_{qt}) = 0. \]

If \( s < k \), then by induction hypothesis, \( a_{0s}b_{00} \in \text{nil}(R) \) and so \( b_{00}a_{0s} \in \text{nil}(R) \). Hence \( b_{00}a_{00}b_{0k} + b_{00}a_{01}b_{0k-1} + \cdots + b_{00}a_{0k-1}b_{01} \in \text{nil}(R) \), since \( R \) is semicommutative. If we multiply \( \sum_{s+t=k} a_{0s}b_{0t} = 0 \) on left side by \( b_{00} \), then we have \( b_{01}a_{0k-1}b_{01} = -(b_{01}a_{00}b_{0k} + b_{01}a_{01}b_{0k-1} + \cdots + b_{01}a_{0k-2}b_{02}) - b_{01}a_{0k}b_{00} = -(b_{01}a_{00})b_{0k} - (b_{01}a_{01})b_{0k-1} - \cdots -(b_{01}a_{0k-2})b_{02} - b_{01}(a_{0k}b_{00}) \in \text{nil}(R) \), since \( R \) is semicommutative. Thus \( a_{0k-1}b_{01} \in \text{nil}(R) \). Similarly, we can show that \( a_{0k-2}b_{02} \in \text{nil}(R), \ldots, a_{00}b_{0k} \in \text{nil}(R) \). So we show that \( a_{is}b_{jt} \in \text{nil}(R) \) for any \( s,t \) with \( s + t = k \) and any \( i,j \) with \( i + j = 0 \). Suppose that \( l \leq p + q \) is such that \( a_{is}b_{jt} \in \text{nil}(R) \) for any \( s,t \) with \( s + t = k \) and any \( i,j \) with \( i + j < l \). We will show that \( a_{is}b_{jt} \in \text{nil}(R) \) for any \( s,t \) with \( s + t = k \) and any \( i,j \) with \( i + j = l \). If \( s < k \), then by induction hypothesis, \( a_{is}b_{00} \in \text{nil}(R) \). So \( b_{00}a_{is} \in \text{nil}(R) \). If \( i < l \), then by induction hypothesis on \( l \), \( a_{ik}b_{00} \in \text{nil}(R) \) for any \( i < l \), and so \( b_{00}a_{ik} \in \text{nil}(R) \) for any \( i < l \). Multiplying \( \sum_{s+t=k} a_{0s}b_{lt} + \sum_{s+t=k} a_{1s}\alpha(b_{(l-1)t}) + \cdots + \sum_{s+t=k} a_{ls}\alpha^l(b_{0t}) = 0 \) on left side by \( b_{00} \), we have \( b_{00}a_{ik}\alpha^l(b_{00}) \in \text{nil}(R) \), since \( \text{nil}(R) \) is an ideal of \( R \). Thus \( b_{00}a_{ik}\alpha^l(b_{00})\alpha^l(a_{ik}) = b_{00}a_{ik}\alpha^l(b_{00}a_{ik}) \in \text{nil}(R) \). Thus \( b_{00}a_{ik} \in \text{nil}(R) \) by Lemma 13, and so \( a_{ik}b_{00} \in \text{nil}(R) \). Similarly, we can show that \( a_{is}b_{jt} \in \text{nil}(R) \) for any \( s,t \) with \( s + t = k \) and any \( i,j \) with \( i + j = l \). Therefore, by induction, we have \( a_{is}b_{jt} \in \text{nil}(R) \) for any \( 0 \leq i \leq p, \) and \( 0 \leq j \leq q \) and any \( s,t \) with \( s + t = 0,1, \ldots, m + n \). Hence \( \sum_{s+t=k} a_{is}b_{jt} \in \text{nil}(R) \), since \( R \) is semicommutative. Thus \( f_i g_j \in \text{nil}(R[x]) \), by Proposition 16. Therefore \( R[x] \) is \( \bar{a} \)-weak Armendariz.

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