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Intuitionistic Fuzzy Rings

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Abstract

In this paper we present a new formulation of intuitionistic fuzzy rings based on the notion of intuitionistic fuzzy space. A relation between intuitionistic fuzzy ring based on intuitionistic fuzzy space and ordinary rings is obtained in terms of induction and correspondence.

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1 Introduction

In 1982 W. J. Liu [8] introduced the concept of fuzzy ring and fuzzy ideal. In 1985 Ren [10] studied the notions of fuzzy ideal and fuzzy quotient ring. Fuzzy rings and fuzzy ideal in the sense of Liu and Ren were actually a rational extension of Rosenfield's fuzzy group by starting with an ordinary ring and then define a fuzzy subring based on the ordinary operations of the given ring.

Based on the notion of fuzzy space which play the role of universal set in ordinary algebra and using fuzzy binary operation K. A. Dib [3] obtained a new formulation for fuzzy rings and fuzzy ideals.

In the case of intuitionistic fuzzy sets there were several attempts to define intuitionistic fuzzy rings [9, 7] by generalizing the approach used by Liu to define fuzzy ring.

In this paper we generalize Dib's notion for fuzzy rings based on the notion of fuzzy space to the case of intuitionistic fuzzy set relying on the notion of intuitionistic fuzzy space and intuitionistic fuzzy function defined by Fathi and Abdul Razak [4, 5, 6] which will serve as a universal set in the classical case.

2 Preliminary Notes

In this section we will recall some of the fundamental concepts and definitions required in the sequel.

Let $L = I \times I$, where $I = [0, 1]$. Define a partial order on L , in terms of the partial order on I, as follows: For every $(r_1, r_2), (s_1, s_2) \in L$

1. $(r_1, r_2) \leq (s_1, s_2)$ iff $r_1 < s_1, r_2 < s_2$ (or $r_1 = s_1$ and $r_2 = s_2$) whenever $s_1 \neq 0 \neq s_2$.

2. $(0, 0) = (s_1, s_2)$ whenever $s_1 = 0 = s_2$.

Thus the cartesian product $L = I \times I$ is a distributive, not complemented lattice. The operation of infimum and supremum in L are given respectively by
$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$ and $(r_1, r_2) \vee (s_1, s_2) = (r_1 \vee s_1, r_2 \vee s_2)$.

Definition 2.1 (Atanassov [1]). *Let X be a nonempty fixed set. An intuitionistic fuzzy set A is an object having the form*

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership and the degree of nonmembership respectively of each element $x \in X$ to the set A, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Remark. The intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ in X will be denoted by $A = \{\langle x, \underline{A}(x), \overline{A}(x)\rangle : x \in X\}$ or simply $A = \big(x, \underline{A}(x), \overline{A}(x)\big)$ where $\underline{A}(x) = \mu_A(x)$ and $\overline{A}(x) = \nu_A(x)$.

The support of the intuitionistic fuzzy set $A = \{\langle x, \underline{A}(x), \overline{A}(x)\rangle : x \in X\}$ in X is the subset $A_\circ$ of X defined by:

$$A_\circ = \{x \in X : \underline{A}(x) \neq 0 \text{ and } \overline{A}(x) \neq 1\}.$$

Definition 2.2 *Let X be a nonempty set. An intuitionistic fuzzy space (simply IFS) denoted by (X, I, I) is the set of all ordered triples (x, I, I), where $(x, I, I) = \{(x, r, s) : r, s \in I$ with $r + s \leq 1$ and $x \in X\}$. the ordered triplet (x, I, I) is called an intuitionistic fuzzy element of the intuitionistic fuzzy space (X, I, I) and the condition $r, s \in I$ with $r + s \leq 1$ will be referred to as the "intuitionistic condition".*

Therefore an intuitionistic fuzzy space is an (ordinary) set with ordered triples. In each triplet the first component indicates the (ordinary) element while the second and the third components indicate its set of possible membership and nonmembership values respectively.

Definition 2.3 *Let $U_\circ$ be a given subset of X. An intuitionistic fuzzy subspace U of the IFS (X, I, I) is the collection of all ordered triples $(x, \underline{u}_x, \overline{u}_x)$, where $x \in U_\circ$ and $\underline{u}_x, \overline{u}_x$ are subsets of I such that $\underline{u}_x$ contains at least one element beside the zero element and $\overline{u}_x$ contains at least one element beside the unit. If $x \notin U_\circ$, then $\underline{u}_x = 0$ and $\overline{u}_x = 1$. The ordered triple $(x, \underline{u}_x, \overline{u}_x)$ will be called an intuitionistic fuzzy element of the intuitionistic fuzzy subspace U. The empty fuzzy subspace denoted by ϕ is defined to be $\phi = \{(x, I, I) : x \in \phi\}$.*

Definition 2.4 *An intuitionistic fuzzy relation ρ from an IFS X to an IFS Y is a subset of the intuitionistic fuzzy cartesian product $(X, I, I)\boxtimes(Y, I, I)$. An intuitionistic fuzzy relation from an IFS X into itself is called an intuitionistic fuzzy relation in the IFS X.*

Definition 2.5 *An intuitionistic fuzzy function between two intuitionistic fuzzy spaces X and Y is an intuitionistic fuzzy relation $\boldsymbol{F}$ from the X to Y satisfying the following conditions:*

1. *For every $x \in X$ with $r, s \in I$, there exists a unique element $y \in Y$ with $w, z \in I$; such that $((x, y),(r, w),(s, z)) \in A$ for some $A \in \boldsymbol{F}$.*

2. *If $((x, y),(r_1, w_1),(s_1, z_1)) \in A \in \boldsymbol{F}$, and $((x, y'),(r_2, w_2),(s_2, z_2)) \in B \in \boldsymbol{F}$ then $y = y'$.*

3. *If $((x, y),(r_1, w_1),(s_1, z_1)) \in A \in\boldsymbol{F}$, and $((x, y),(r_2, w_2),(s_2, z_2)) \in B \in\boldsymbol{F}$ then $(r_1 > r_2)$ implies $(w_1 > w_2)$ and $(s_1 > s_2)$ implies $(z_1 < z_2)$.*

4. *If $((x, y),(r, w),(s, z)) \in A \in\boldsymbol{F}$, then $r = 0$ implies $w = 0$, $s = 1$ implies $z = 0$, and $r = 1$ implies $w = 1$, $s = 0$ implies $z = 1$.*

Thus conditions (1) and (2) imply that there exists a unique (ordinary) function from X to Y, namely $F : X \to Y$ and that for every $x \in X$ there exists unique (ordinary) functions from I to I, namely $\underline{f}_x, \overline{f}_x : I \to I$. On the other hand conditions (3) and (4) are respectively equivalent to the following conditions:
(i) $\underline{f}_x$ is nondecreasing on I and $\overline{f}_x$ is nonincreasing on I.
(ii) $\underline{f}_x(0) = 0 = \overline{f}_x(1)$ and $\underline{f}_x(1) = 1 = \overline{f}_x(0)$.

That is, an intuitionistic fuzzy function between two intuitionistic fuzzy spaces X and Y is a function $\boldsymbol{F}$ from X to Y characterized by the ordered triple

$$\left(F(x), \{\underline{f}_x\}_{x\in X}, \{\overline{f}_x\}_{x\in X}\right),$$

where $F(x)$ is a function from X to Y and $\{\underline{f}_x\}_{x\in X}, \{\overline{f}_x\}_{x\in X}$ are family of functions from I to I satisfying the conditions (i) and (ii) such that the image of any intuitionistic fuzzy subset A of the IFS X under $\boldsymbol{F}$ is the intuitionistic fuzzy subset $\boldsymbol{F}(A)$ of the IFS Y defined by

$$\boldsymbol{F}(A)y = \begin{cases} \left(\bigvee_{x \in F^{-1}(y)} \underline{f}_x(\mu_A(x)), \bigwedge_{x \in F^{-1}(y)} \overline{f}_x(\nu_A(x)) \right) & \text{if } F^{-1}(y) \neq \phi \\ (0,1) & \text{if } F^{-1}(y) = \phi \end{cases}$$

We will call the functions $\underline{f}_x, \overline{f}_x$ the *comembership functions* and the *cononmembership functions* respectively. The intuitionistic fuzzy function $\boldsymbol{F}$ will be denoted by $\boldsymbol{F} = (F, \underline{f}_x, \overline{f}_x)$.

Definition 2.6 *An intuitionistic fuzzy binary operation* $\boldsymbol{F}$ *on an IFS* (X, I, I) *is an intuitionistic fuzzy function* $\boldsymbol{F} : X \times X \to X$ *with comembership functions* $\underline{f}_{xy}$ *and cononmembership functions* $\overline{f}_{xy}$ *satisfying:*

1. $\underline{f}_{xy}(r,s) \neq 0$ *iff* $r \neq 0$ *and* $s \neq 0$, *and* $\overline{f}_{xy}(w,z) \neq 1$ *iff* $w \neq 1$ *and* $z \neq 1$.

2. $\underline{f}_{xy}, \overline{f}_{xy}$ *are onto. That is,* $\underline{f}_{xy}(I \times I) = I$ *and* $\overline{f}_{xy}(I \times I) = I$.

Thus for any two intuitionistic fuzzy elements $(x, I, I), (y, I, I)$ of the IFS X and any intuitionistic fuzzy binary operation $\boldsymbol{F} = \left(F, \underline{f}_{xy}, \overline{f}_{xy}\right)$ defined on an IFS X. The action of the intuitionistic fuzzy binary operation $\boldsymbol{F}$ over the IFS X is given by

$$\begin{aligned} (x, I, I)\boldsymbol{F}(y, I, I) &= \boldsymbol{F}((x, I, I), (y, I, I)) \\ &= (F(x,y), \underline{f}_{xy}(I \times I), \overline{f}_{xy}(I \times I)) \\ &= (F(x,y), I, I). \end{aligned}$$

An intuitionistic fuzzy binary operation is said to be *uniform* if the associated comembership and cononmembership functions are identical. That is, if $\underline{f}_{xy} = \underline{f}, \overline{f}_{xy} = \overline{f}$ for all $x, y \in X$. A left semiuniform (right semiuniform) fuzzy binary operation is an intuitionistic fuzzy binary operation having identical comembership functions (cononmembership functions).

Definition 2.7 *An intuitionistic fuzzy groupoid, denoted by* $((X, I, I), \boldsymbol{F})$, *is an IFS* (X, I, I) *together with an intuitionistic fuzzy binary operation* $\boldsymbol{F}$ *defined over it. An intuitionistic fuzzy semigroup is an intuitionistic fuzzy groupied that is associative. An intuitionistic fuzzy monoid is an intuitionistic fuzzy semigroup that admits an identity. An intuitionistic fuzzy group is an intuitionistic fuzzy monoid in which each intuitionistic fuzzy element has an inverse.*

An intuitionistic fuzzy group $((G, I, I), \boldsymbol{F})$ is called an abelian (commutative) intuitionistic fuzzy group if and only if for all $(x, I, I), (y, I, I) \in ((G, I, I), \boldsymbol{F})$

$$(x, I, I)\boldsymbol{F}(y, I, I) = (y, I, I)\boldsymbol{F}(x, I, I).$$

3 Intuitionistic Fuzzy Ring

We will define intuitionistic fuzzy ring by adding two intuitionistic fuzzy binary operations to a given fuzzy space with similar conditions to the ordinary case.

Definition 3.1 *An intuitionistic fuzzy ring, denoted by* $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$, *is an IFS* (R,I,I) *together with two intuitionistic fuzzy binary operations, namely* $\boldsymbol{F}^{+},\boldsymbol{F}^{*}$ *satisfying the following conditions*

(1) $\big((R,I),\boldsymbol{F}^{+}\big)$ *is an abelian intuitionistic fuzzy group,*

(2) $((R,I),\boldsymbol{F}^{*})$ *is an intuitionistic fuzzy semi-group,*

(3) $\boldsymbol{F}^{*}$ *is distributive over* $\boldsymbol{F}^{+}$. *That is,*

$$(x,I,I)\boldsymbol{F}^{*}\big((y,I,I)\boldsymbol{F}^{+}(z,I,I)\big)=(x,I,I)\boldsymbol{F}^{*}(y,I,I)\boldsymbol{F}^{+}(x,I,I)\boldsymbol{F}^{*}(z,I,I),$$

$$\big((x,I,I)\boldsymbol{F}^{+}(y,I,I)\big)\,\boldsymbol{F}^{*}(z,I,I)=(x,I,I)\boldsymbol{F}^{*}(z,I,I)\boldsymbol{F}^{+}(y,I,I)\boldsymbol{F}^{*}(z,I,I).$$

A uniform (left uniform, right uniform) intuitionistic fuzzy ring is an intuitionistic fuzzy ring with uniform (left uniform, right uniform) intuitionistic fuzzy binary operations. A commutative (abelian) intuitionistic fuzzy ring is an intuitionistic fuzzy ring whose elements are commutive with respect to the intuitionistic fuzzy binary operation $\boldsymbol{F}^{*}$. By the order of an intuitionistic fuzzy ring we mean the number of intuitionistic fuzzy elements in the intuitionistic fuzzy ring. An intuitionistic fuzzy ring of infinite order is an infinite intuitionistic fuzzy ring.

Definition 3.2 $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *be an intuitionistic fuzzy ring.*
(1) *An intuitionistic fuzzy element in* $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *is an intuitionistic fuzzy unity and denoted by* $(1,I,I)$ *if* $(1,I,I)\boldsymbol{F}^{*}(b,I,I)=(b,I,I)\boldsymbol{F}^{*}(1,I,I)=(b,I,I)$ *for all* $(b,I,I)\in\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$. *An intuitionistic fuzzy ring having a unity is called an intuitionistic fuzzy ring with unity.*
(2) *An intuitionistic fuzzy element* $(a,I,I)\in\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *is called a unit if there exist an intuitionistic fuzzy element* (b,I,I) *in* $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *such that* $(a,I,I)\boldsymbol{F}^{*}(b,I,I)=(b,I,I)\boldsymbol{F}^{*}(a,I,I)=(1,I,I)$.

Remark. For the intuitionistic fuzzy ring $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ we will call $\boldsymbol{F}^{+}$ and $\boldsymbol{F}^{*}$ the additive and multiplicative intuitionistic fuzzy binary operation respectively. The intuitionistic fuzzy identity will be denoted by $(0,I,I)$ and for any intuitionistic fuzzy element $(a,I,I)\in\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ the additive and multiplicative intuitionistic fuzzy inverse will be denoted by $(-a,I,I)$ and (a^{-1},I,I) respectively. That is;

$(a,I,I)\boldsymbol{F}^{+}(-a,I,I)=(-a,I,I)\boldsymbol{F}^{+}(a,I,I)=(0,I,I),$

$(a,I,I)\boldsymbol{F}^{*}(a^{-1},I,I)=(a^{-1},I,I)\boldsymbol{F}^{*}(a,I,I)=(1,I,I).$

The next theorem gives a correspondence relation between intuitionistic fuzzy rings and both ordinary and fuzzy rings.

Theorem 3.3 (1) *Associated to each intuitionistic fuzzy ring* $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *where* $\boldsymbol{F}^{+}=\Big(F^{+},\underline{f}^{+}_{xy},\overline{f}^{+}_{xy}\Big)$ *and* $\boldsymbol{F}^{*}=\Big(F^{*},\underline{f}^{*}_{xy},\overline{f}^{*}_{xy}\Big)$ *two fuzzy rings* $\big((R,I),\underline{\boldsymbol{F}}^{+},\underline{\boldsymbol{F}}^{*}\big)$ *and* $\Big((R,I),\overline{\boldsymbol{F}^{+}},\overline{\boldsymbol{F}^{*}}\Big)$ *where* $\underline{\boldsymbol{F}}^{+}=(F^{+},\underline{f}^{+}_{xy})$, $\underline{\boldsymbol{F}}^{*}=(F^{*},\underline{f}^{*}_{xy})$ *and* $\overline{\boldsymbol{F}^{+}}=(F^{+},1-\overline{f}^{+}_{xy})$, $\overline{\boldsymbol{F}^{*}}=(F^{*},1-\overline{f}^{*}_{xy})$ *which are isomorphic to the intuitionistic fuzzy ring* $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *by the correspondence* $(x,I,I)\leftrightarrow(x,I):x\in R$.
(2) *To each intuitionistic fuzzy ring* $\big((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*}\big)$ *there is an associated (ordinary) ring* (R,F^{+},F^{*}) *which is isomorphic to the intuitionistic fuzzy ring by the correspondence* $(x,I,I)\leftrightarrow x:x\in R$.

Example 3.4 (1) Consider the set $R=\{a\}$. Define the intuitionistic fuzzy binary operations $\boldsymbol{F}^{+}=(F,\underline{f}^{+}_{xy},\overline{f}^{+}_{xy})$, $\boldsymbol{F}^{*}=(F,\underline{f}^{*}_{xy},\overline{f}^{*}_{xy})$ over the intuitionistic fuzzy space (R,I,I) such that:
$F(a,a)=a$ and $\underline{f}_{aa}(r,s)=r\wedge s$, $\overline{f}_{aa}(r,s)=r\vee s$.
$G(a,a)=a$ and $\underline{g}_{aa}(r,s)=r\vee s$, $\overline{f}_{aa}(r,s)=r\wedge s$.
Thus, the intuitionistic fuzzy space (R,I,I) together with $\boldsymbol{F}^{+},\boldsymbol{F}^{*}$ define a (trivial) intuitionistic fuzzy ring $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$.
(2) Consider the set $\mathbb{Z}_3=\{0,1,2\}$. Define the intuitionistic fuzzy binary operations $\boldsymbol{F}^{+}=(F,\underline{f}^{+}_{xy},\overline{f}^{+}_{xy})$ and $\boldsymbol{F}^{*}=(F,\underline{f}^{*}_{xy},\overline{f}^{*}_{xy})$ over the intuitionistic fuzzy space $(\mathbb{Z}_3,I,I)$ as follows:
$F^{+}(x,y)=x+_3 y$, where $+_3$ refers to addition modulo 3,
and $\underline{f}^{+}_{xy}(r,s)=r\cdot s$, $\overline{f}^{+}_{xy}(r,s)=r\cdot s$.
$F^{*}(x,y)=x\times_3 y$, where $\times_3$ refers to multiplication modulo 3,
and $\underline{f}^{*}_{xy}(r,s)=r\cdot s$, $\overline{f}^{*}_{xy}(r,s)=r\cdot s$.
Thus $((\mathbb{Z}_3,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is an intuitionistic fuzzy ring.

Definition 3.5 *Let* S *be an intuitionistic fuzzy subspace of the intuitionistic fuzzy space* (R,I,I). *The ordered pair* $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ *is an intuitionistic fuzzy subring of the intuitionistic fuzzy ring* $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ *if and only if* $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ *defines an intuitionistic fuzzy ring under the intuitionistic fuzzy binary operations* $\boldsymbol{F}^{+},\boldsymbol{F}^{*}$.

Obviously, if $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is an intuitionistic fuzzy subring of $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ and $(T;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is an intuitionistic fuzzy subring of $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$, then $(T;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is an intuitionistic fuzzy subring of $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$. Also if $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is an IFR with an intuitionistic fuzzy identity (e,I,I) then both $((e,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ and $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ are (trivial) intuitionistic fuzzy subrings of $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$.

Theorem 3.6 *Let* $S=\{(x,\underline{s}_x,\overline{s}_x:x\in S_\circ\}$ *be an intuitionistic fuzzy subspace of the intuitionistic fuzzy space* (R,I,I). *Then* $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ *is an intuitionistic fuzzy subring of the IFR* $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ *iff:*

(1) $(S_\circ; F, G)$ *is an (ordinary) subring of the ring* (R, F^+, F^*),
(2) *for all* $x, y \in S_\circ$ *we have*

$$\underline{s}_x \underline{f}^+_{xy} \underline{s}_y = \underline{s}_{xF^+y}\,, \quad \overline{s}_x \overline{f}^+_{xy} \overline{s}_y = \overline{s}_{xF^+y}$$

$$\underline{s}_x \underline{f}^*_{xy} \underline{s}_y = \underline{s}_{xF^*y}\,, \quad \overline{s}_x \overline{f}^*_{xy} \overline{s}_y = \overline{s}_{xF^*y}.$$

Proof. Suppose (1) and (2) are satisfied, then:
The intuitionistic fuzzy subspace S is closed under the intuitionistic fuzzy binary operations $\boldsymbol{F}^+$ and $\boldsymbol{F}^*$: Let $(x, \underline{s}_x, \overline{s}_x), (y, \underline{s}_y, \overline{s}_y)$ be in S then

$$\begin{aligned}(x, \underline{s}_x, \overline{s}_x)\boldsymbol{F}^+(y, \underline{s}_y, \overline{s}_y) &= \boldsymbol{F}^+((x, \underline{s}_x, \overline{s}_x), (y, \underline{s}_y, \overline{s}_y)) \\ &= (F^+(x, y), \underline{f}^+_{xy}(\underline{s}_x, \underline{s}_y), \overline{f}^+_{xy}(\overline{s}_x, \overline{s}_y))) \\ &= ((xF^+y), \underline{s}_{xF^+y}, \overline{s}_{xF^+y}) \\ &\in S.\end{aligned}$$

Similarly, for the intuitionistic fuzzy binary operation $\boldsymbol{F}^*$

$$\begin{aligned}(x, \underline{s}_x, \overline{s}_x)\boldsymbol{F}^*(y, \underline{s}_y, \overline{s}_y) &= \boldsymbol{F}^*((x, \underline{s}_x, \overline{s}_x), (y, \underline{s}_y, \overline{s}_y)) \\ &= (F^*(x, y), \underline{f}^*_{xy}(\underline{s}_x, \underline{s}_y), \overline{f}^*_{xy}(\overline{s}_x, \overline{s}_y))) \\ &= ((xF^*y), \underline{s}_{xF^*y}, \overline{s}_{xF^*y}) \\ &\in S.\end{aligned}$$

From (1) and since $(S_\circ; F^+, F^*)$ is an (ordinary) subring of the ring (G, F^+, F^*) it is easy to check that $(S; \boldsymbol{F}^+, \boldsymbol{F}^*)$ is an intuitionistic fuzzy subring of the IFR $((G, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$

Conversely if $(S; \boldsymbol{F}^+, \boldsymbol{F}^*)$ is an intuitionistic fuzzy subring of $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ then (1) holds by the associativity theorem. Also the following hold

$$\underline{s}_x \underline{f}^+_{xy} \underline{s}_y = \underline{f}^+_{xy}(\underline{s}_x \times \underline{s}_y) = \underline{s}_{xF^+y}$$

$$\overline{s}_x \overline{f}^+_{xy} \overline{s}_y = \overline{f}^+_{xy}(\overline{s}_x \times \overline{s}_y) = \overline{s}_{xF^+y}$$

$$\underline{s}_x \underline{f}^*_{xy} \underline{s}_y = \underline{f}^*_{xy}(\underline{s}_x \times \underline{s}_y) = \underline{s}_{xF^*y}$$

$$\overline{s}_x \overline{f}^*_{xy} \overline{s}_y = \overline{f}^*_{xy}(\overline{s}_x \times \overline{s}_y) = \overline{s}_{xF^*y}$$

being onto comembership and cononmembership functions over the partial ordered sub-lattices $\underline{s}_x \times \underline{s}_y$ and $\overline{s}_x \times \overline{s}_y$ respectively of the vector lattice $I \times I$.

Example 3.7 Let $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ be defined as in Example 4.2.5 (1). Consider the intuitionistic fuzzy subspace $S = \{(a, [0, \alpha], [\beta, 1])\}$ such that $0 < \alpha < \beta < 1$. It is easy to check that $(S, \boldsymbol{F}^+, \boldsymbol{F}^*)$ is an intuitionistic fuzzy subring of the intuitionistic fuzzy ring $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$. Now consider the intuitionistic fuzzy subspace $S' = \{(a, \{0\}, \{1\})\}$, then $(S', \boldsymbol{F}^+, \boldsymbol{F}^*)$ defines another intuitionistic fuzzy subring of the intuitionistic fuzzy ring $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ which is different from the case of ordinary rings, since an ordinary trivial ring have only one subring, namely the ring itself.
(2) Let $((\mathbb{Z}_3, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ be defined as in Example 4.2.5 (2). Consider the intuitionistic fuzzy subspace $Z = \{(0, [0, \alpha], [\beta, 1]), (1, [0, \gamma], [\delta, 1])\}$ such that $0 < \alpha < \beta < 1$ and $0 < \gamma < \delta < 1$. We have $(Z, \boldsymbol{F}^+, \boldsymbol{F}^*)$ is not an intuitionistic fuzzy subring of $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$, since Z is not closed under $\boldsymbol{F}^+$. For instance, $(0, [0, \alpha], [\beta, 1])\boldsymbol{F}(1, [0, \gamma], [\delta, 1]) = (1, [0, \alpha \cdot \gamma], [\beta \cdot \delta, 1]) \notin Z$.
If $\alpha, \gamma = 1$ and $\beta, \delta = 0$, then $Z = \{(0, I, I), (1, I, I)\}$ together with $\boldsymbol{F}^+, \boldsymbol{F}^*$ defines an intuitionistic fuzzy subring of $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$.

Now we introduce the notions of Intuitionistic fuzzy integral domain and intuitionistic fuzzy field.

Definition 3.8 *(1) An intuitionistic fuzzy zero-divisor is an intuitionistic fuzzy element* $(a, I, I) \neq (0, I, I)$ *of a commutative intuitionistic fuzzy ring* $((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ *such that there is an intuitionistic fuzzy element* $(b, I, I) \in ((R, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ *where* $(b, I, I) \neq (0, I, I)$ *with* $(a, I, I)\boldsymbol{F}^*(b, I, I) = (0, I, I)$.
(2) An intuitionistic fuzzy integral domain is a commutative intuitionistic fuzzy ring with unity and no intuitionistic fuzzy zero divisor.

Example 3.9 (1) Let $((\mathbb{Z}_3, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ be defined as in Example 4.2.5 (2). It is easy to check that $((\mathbb{Z}_3, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ is an intuitionistic fuzzy integral domain with unity $(1, I, I)$ and intuitionistic fuzzy identity $(0, I, I)$.
(2) Consider the set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$. Define the intuitionistic fuzzy binary operations $\boldsymbol{F}^+ = (F, \underline{f}^+_{xy}, \overline{f}^+_{xy})$ and $\boldsymbol{F}^* = (F, \underline{f}^*_{xy}, \overline{f}^*_{xy})$ over the intuitionistic fuzzy space $(\mathbb{Z}_4, I, I)$ as follows:

$F^+(x, y) = x +_4 y$, where $+_4$ refers to addition modulo 4,
and $\underline{f}^+_{xy}(r, s) = r \cdot s$, $\overline{f}^+_{xy}(r, s) = r \cdot s$.
$F^*(x, y) = x \times_4 y$, where $\times_4$ refers to multiplication modulo 4,
and $\underline{f}^*_{xy}(r, s) = r \cdot s$, $\overline{f}^*_{xy}(r, s) = r \cdot s$.

Thus $((\mathbb{Z}_4, I, I), \boldsymbol{F}^+, \boldsymbol{F}^*)$ is an commutative intuitionistic fuzzy ring with unity $(1, 0, 0)$ which is not an intuitionistic fuzzy integral domain since $(2, I, I)\boldsymbol{F}^*(2, I, I) = (0, I, I)$.

The next theorem introduce the *cancellation property* for intuitionistic fuzzy elements in a given intuitionistic fuzzy integral domain.

Theorem 3.10 *Let (a,I,I), (b,I,I) and (c,I,I) belong to an intuitionistic fuzzy integral domain $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$. If $(a,I,I)\neq(0,I,I)$ and $(a,I,I)\boldsymbol{F}^{*}(b,I,I)=(a,I,I)\boldsymbol{F}^{*}(c,I,I)$, then $(b,I,I)=(c,I,I)$.*

Proof. From $(a,I,I)\boldsymbol{F}^{*}(b,I,I)=(a,I,I)\boldsymbol{F}^{*}(c,I,I)$ we have:
$(a,I,I)\boldsymbol{F}^{*}\left((b,I,I)\boldsymbol{F}^{+}(-c,I,I)\right)=(0,I,I)$.
Since $(a,I,I)\neq(0,I,I)$ we must have:
$(b,I,I)\boldsymbol{F}^{+}(-c,I,I)$. That is $(b,I,I)=(c,I,I)$.

A necessary and sufficient condition for intuitionistic fuzzy rings in terms of the cancellation property is given the next corollary which is a direct result from the above theorem.

Corollary 3.11 *A commutative intuitionistic fuzzy ring with unity is an intuitionistic fuzzy integral domain iff the cancellation property holds.*

Definition 3.12 *An intuitionistic fuzzy field is a commutative intuitionistic fuzzy ring $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ with unity in which every intuitionistic fuzzy element $(a,I,I)\neq(0,I,I)$ in $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is a unit.*

Example 3.13 Consider the intuitionistic fuzzy ring $((\mathbb{Z}_3,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ defined in Example 4.2.5 (2). It is easy to check that $((\mathbb{Z}_3,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is an intuitionistic fuzzy field. If we replace the intuitionistic fuzzy space $(\mathbb{Z}_3,I,I)$ with the intuitionistic fuzzy space $(\mathbb{Z},I,I)$ then the outcome will be a commutative intuitionistic fuzzy ring $((\mathbb{Z},I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ which is an intuitionistic fuzzy integral domain but not an intuitionistic fuzzy field.

Now we move further to define intuitionistic fuzzy ideal which is expected to be analogous to intuitionistic fuzzy normal subgroup.

Definition 3.14 *Let $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ be an intuitionistic fuzzy ring. An intuitionistic fuzzy subring $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ of $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ having the property $(a,I,I)\boldsymbol{F}^{*}(b,I,I)\in(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ for all $(a,I,I)\in((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ and $(b,I,I)\in(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is called left intuitionistic fuzzy ideal in $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$. $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is called a right intuitionistic fuzzy ideal if it satisfies the property $(b,I,I)\in(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ for all $(a,I,I)\in((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ and $(b,I,I)\in(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$. $(S;\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ is called an intuitionistic fuzzy ideal if it is both right and left intuitionistic fuzzy ideal.*

One can easily check that $\{(0,I,I)\}$ and $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ are both left and right intuitionistic fuzzy ideals in $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$. Therefore we will call $\{(0,I,I)\}$ and $((R,I,I),\boldsymbol{F}^{+},\boldsymbol{F}^{*})$ an improper intuitionistic fuzzy ideals.

The next theorem is a straightforward result to ideals obtained by intuitionistic fuzzy elements in an intuitionistic fuzzy ring.

Theorem 3.15 *Let* $((R, I, I), \boldsymbol{F}^{+}, \boldsymbol{F}^{*})$ *be a commutative intuitionistic fuzzy ring having an intuitionistic fuzzy identity* $(0, I, I)$ *and let* (p, I, I) *be an arbitrary intuitionistic fuzzy element in* $((R, I, I), \boldsymbol{F}^{+}, \boldsymbol{F}^{*})$ *then*
(1) $P = \{(p, I, I)\boldsymbol{F}^{*}(r, I, I) : (r, I, I) \in ((R, I, I), \boldsymbol{F}^{+}, \boldsymbol{F}^{*})\}$ *is an intuitionistic fuzzy ideal in* $((R, I, I), \boldsymbol{F}^{+}, \boldsymbol{F}^{*})$.
(2) $T = \{(r, I, I) : (r, I, I)\boldsymbol{F}^{*}(t, I, I) = (0, I, I) : (r, I, I), (t, I, I) \in ((R, I, I), \boldsymbol{F}^{+}, \boldsymbol{F}^{*})\}$ *is an intuitionistic fuzzy ideal in* $((R, I, I), \boldsymbol{F}^{+}, \boldsymbol{F}^{*})$.

4 Conclusion

In this paper, we have generalized the study initiated in [3] about fuzzy rings to the context of intuitionistic fuzzy rings. In the absence of the intuitionistic fuzzy universal set, formulation of the intrinsic definition for an intuitionistic fuzzy subgroup is not evident. In this paper we define the notion of an intuitionistic fuzzy ring using the notion of an intuitionistic fuzzy space. The use of intuitionistic fuzzy space as a universal set corrects the deviation in the definition of intuitionistic fuzzy subgroups. This concept can be considered as a new formulation of the classical theory of intuitionistic fuzzy rings obtained in [9, 7].

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