On Strongly Clean Rings

V. A. Hiremath

Visiting professor, Department of Mathematics
Manglore University, Mangalagangotri-574199, India
va_hiremath@rediffmail.com

Sharad Hegde

Department of Mathematics, Karnatak University
Dharwad-580003, India
sharum_kud@yahoo.co.in

Abstract

A ring \( R \) is said to be strongly clean if every element of \( R \) is sum of an idempotent and a unit in \( R \) which commute with each other. A ring \( R \) is said to have stable range one if for any \( a, b \) in \( R \) with \( aR + bR = R \) there exists \( y \) in \( R \) such that \( a + by \) is a unit. In this paper we give a partial answer to the following open problem:

Does every strongly clean ring have stable range one?

Also we give a necessary and sufficient condition for \( R(M) \) (the idealization of \( R \) and \( M \) where \( M \) is an \( R \)-module) to be an exchange ring and to have stable range one.

Mathematics Subject Classification: 16E50

Keywords: exchange ring, strongly clean ring

Introduction

Throughout this paper, rings we consider are associative with identity. If \( R \) is a ring \( U(R) \) and \( Id(R) \) respectively denote the set of all units and all idempotents in \( R \).

A ring \( R \) is said to be an exchange ring if for each \( a \in R \) there exists \( e \in Id(R) \) such that \( e \in aR \) and \( (1 - e) \in (1 - a)R \). The notion of a clean ring was introduced by Nicholson[6] in his study of lifting idempotents and exchange rings. He defined a ring \( R \) to be clean if every element can be written

\[ a = xe + y(1 - e) \]

where \( x, y \in U(R) \) and \( e \in Id(R) \).
as a sum of an idempotent and a unit. Nicholson[5] defined an element $a$ of a ring $R$ to be strongly clean if $a = e + u$ where $e \in Id(R)$, $u \in U(R)$ and $eu = ue$ and the ring $R$ is said to be strongly clean if every element of $R$ is strongly clean. A ring $R$ is said to be strongly $\pi$-regular if for every element $a \in R$ there exists $x \in R$ and $n \in \mathbb{N}$ such that $a^n = a^{n+1}x$. It is well known that: Strongly $\pi$-regular $\Rightarrow$ Strongly clean $\Rightarrow$ Clean $\Rightarrow$ Exchange

A ring $R$ is said to have stable range one if for any $a, b \in R$ with $aR + bR = R$ there exists $y \in R$ such that $a + by \in U(R)$. In 1996, P.Ara[2] proved that strongly $\pi$-regular ring has stable range one. So, Nicholson[5] asked the following question: Does every strongly clean ring have stable range one?

In this paper we give a partial answer to the above open problem. Also we give a necessary and sufficient condition for $R(M)$- (the idealization of $R$ and $M$ where $M$ is an $R$-module)-to be an exchange ring and to have stable range one.

We start with the following definition.

**Definition 1:** A ring $R$ is said to be uniquely $p$-semipotent if every non-trivial principal right ideal $I$ contains unique non-zero idempotent. Equivalently, a ring $R$ is said to be uniquely $p$-semipotent if for every non-zero, non-right invertible $a \in R$ there exists a unique non-zero idempotent $e \in R$ such that $e \in aR$.

**Example 2:** Every division ring is uniquely $p$-semipotent.

**Example 3:** If $D$ is a division ring then $R = D \times D$ is uniquely $p$-semipotent.

Remark: Direct product of uniquely $p$-semipotent rings need not be uniquely $p$-semipotent. We prove this as a corollary to the following result.

**Proposition 4:** If $R$ is a uniquely $p$-semipotent ring with a non-trivial central idempotent then $R$ contains at most four idempotents and is abelian.

**Proof:** Let $e$ be a non-trivial central idempotent in $R$. Let $f^2 = f \in R$. Now we prove that $f \in \{0, 1, e, 1-e\}$. Clearly $ef$ is an idempotent in $eR$. (because $e$ is central) Since $R$ is a uniquely $p$-semipotent ring, $ef = 0$ or $ef = e$.

Case 1: $ef = 0$. Then $f = (1-e)f \in (1-e)R \Rightarrow f = 0$ or $f = 1-e$(because $R$ is uniquely $p$-semipotent).

Case 2: $ef = e$. Then $1 - f = (1-e)(1-f) \in (1-e)R \Rightarrow 1 - f = 0$ or $1 - f = 1 - e$ (because $R$ is uniquely $p$-semipotent).

Therefore in any case $f \in \{0, 1, e, 1-e\}$. Since $e$ is central, so is $1-e$ and hence $R$ is abelian.

**Corollary 5:** Direct product of uniquely $p$-semipotent rings need not be uniquely $p$-semipotent.

**Proof:** Let $R = \prod_{i=1}^{n} D_i$ ($n \geq 3$) where each $D_i$ is a division ring. Since each $D_i$ is abelian ring we get that $R$ is abelian. Also note that $R$ contains more than four idempotents (in fact $R$ contains $2^n$ idempotents). Hence by above proposition $R$ is not uniquely $p$-semipotent.

**Remark:** Strongly clean ring need not be uniquely $p$-semipotent.
For, let $R$ be a commutative local ring which is not a field (i.e. a local ring containing at least one non-zero, non-unit). Then clearly $R$ is a strongly clean ring but not uniquely p-semipotent.

**Theorem 6:** If $R$ is uniquely p-semipotent ring then for each $e^2 = e \in R$, $eRe$ is uniquely p-semipotent ring.

**Proof:** Let $x \in eRe$ be such that $xeRe$ is a non-trivial principal right ideal of $eRe$. Then clearly $xR$ is a non-trivial principal right ideal of $R$. This implies, by hypothesis, that there exists a unique $0 \neq f = f^2 \in R$ such that $f \in xR \Rightarrow f = exer$ for some $r \in R$(because $x \in eRe \Rightarrow x = exe$) \Rightarrow ef = f \Rightarrow fe = f(1)$

Thus $(fe)^2 = fefe = fe$. Clearly $fe \neq 0$(because if $fe = 0$ then by (1), $f = 0$ which is a contradiction). Clearly $fe \in xeRe$. Suppose $0 \neq g = g^2 \in xeRe \Rightarrow 0 \neq g^2 = g \in xR \Rightarrow (by \ hypothesis) g = f \Rightarrow g = ge = fe$. Therefore $fe$ is the unique non-zero idempotent in $xeRe$. Hence $eRe$ is uniquely p-semipotent ring.

Recall that an element $x$ in $R$ is said to be regular (in the sense of von-Neumann) if there exists an element $y$ in $R$ such that $x = xyx$ and $x$ is said to be unit regular if $x = xux$ for some $u \in U(R)$. Camillo and Yu [3, Theorem 3] have proved that an exchange ring $R$ has stable range one if and only if every regular element of $R$ is unit regular. Since strongly clean ring $R$ is an exchange ring, the Camillo and Yu’s result motivated us to find out regular elements of $R$ and the following proposition is helpful in that direction.

**Proposition 7:** Let $R$ be a uniquely p-semipotent, strongly clean ring and let $aR$ be a non-trivial principal right ideal where $a = 1 - e + u$ with $e \neq 0$ is a strongly clean expression of $a$. Then $a$ is a regular element of $R$.

**Proof:** We have, $a = 1 - e + u \Rightarrow ae = ae = eu \Rightarrow ae^{-1} = e \Rightarrow e \in aR(1)$

Now we prove that $a \in eR$. Clearly $a(1 - e)R \subseteq aR(2)$

We prove that $a(1 - e) = 0$. Suppose that $a(1 - e) \neq 0$. Then clearly $a(1 - e)R$ is non-trivial principal right ideal of $R$ and since $R$ is uniquely p-semipotent, $a(1 - e)R$ contains a unique non-zero idempotent $f$. But by (1) and (2) $f = e \Rightarrow e = a(1 - e)r$ for some $r \in R \Rightarrow e = (1 - e)ar$ (because $a(1 - e) = (1 - e)a \Rightarrow e = e^2 = e(1 - e)ar = 0$ which is a contradiction.

Therefore $a(1 - e) = 0 \Rightarrow (1 - e)a = 0 \Rightarrow a = ea \Rightarrow a \in eR(3)$. So, by (1) and (3), $aR = eR$. Hence $a$ is a regular element of $R$.

**Corollary 8:** Let $R$ be a uniquely p-semipotent, strongly clean ring in which no non-zero quasi-regular element is regular. Then $R$ has stable range one.

**Proof:** Let $a$ be a nonzero element of $R$ which is not quasi-regular. Then $a$ has a strongly clean expression $a = 1 - e + u$ with $e \neq 0$. We prove that $a$ is regular. If $a$ is right invertible then clearly $a$ is regular. So suppose that $aR$ is a non-trivial principal right ideal of $R$. Then by above proposition $a$ is regular with $aR = eR$. This implies that $a = ea = e(1 - e + u) = eu \Rightarrow au^{-1}a = euu^{-1}a = ea = a$. So $a$ is unit regular and hence the result follows.
Remark: In a ring with identity, for any element $e$ of $R$.

Proposition 9: Let $R$ be a ring. If $e^2 = e \in R$ then $e$ is USC.

Proof: Let $e^2 = e \in R$. We know that $1 - e \in Id(R)$ and $2e - 1$ is a unit with $(2e - 1)^{-1} = (2e - 1)$. So $e = (1 - e) + (2e - 1)$ is a strongly clean expression of $e$. Suppose $e = f + u$ is another strongly clean expression. Then we have $(e - f)(e + f) = e + ef - fe - f = e - f$ (because $ef = fe$) $\Rightarrow e + f = 1$(because $e - f$ is a unit)$\Rightarrow f = 1 - e$. Hence $e$ is USC.

Remark: In a ring with identity, for any $e^2 = e \in R$, $-e$ is always a strongly clean element, but it need not be USC. For, $-e = (1 - e) + (-1)$ is a strongly clean expression. But in any field $F$ with $\text{char}(F) \neq 2$, $-1$ has two different strongly clean expressions namely $-1 = 0 + (-1)$ and $-1 = 1 + (-2)$.

Proposition 10: For a ring $R$ the following are equivalent:

1) $R$ is strongly clean.
2) Every element $x \in R$ can be written as $x = u - e$ where $u \in U(R), e \in Id(R)$ and $ue = eu$.
3) Every element $x \in R$ can be written as $x = u + e$ where $u \in U(R) \cup \{0\}, e \in Id(R)$ and $ue = eu$.
4) Every element $x \in R$ can be written as $x = u - e$ where $u \in U(R) \cup \{0\}, e \in Id(R)$ and $ue = eu$.

Proof: 1) $\Rightarrow$ 2). Let $x \in R$. Let $-x = e + u$ be a strongly clean expression of $-x$. Then we have $x = -u - e$ where $-u \in U(R), e \in Id(R)$ and $(-u)e = e(-u)$. 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 4) are similar to 1) $\Rightarrow$ 2).

4) $\Rightarrow$ 1). Let $x \in R$ $\Rightarrow$ (by hypothesis) $-x = u - e$ for some $u \in U(R) \cup \{0\}, e \in Id(R)$ and $eu = ue$ $\Rightarrow x = e + (-u)$, the case when $u = 0$ follows from proposition 2.9.

Proposition 11: Let $R$ be a ring. Then $R = U(R) \cup Id(R)$ if and only if $R$ is a division ring or $R$ is a Boolean ring.

Proof: Suppose $R = U(R) \cup Id(R)$. If $R$ is a Boolean ring we are done. So suppose that there exists an element $u$ in $R$ such that $u \notin Id(R)$. This implies, by hypothesis, $u \in U(R)$. We claim that every non-zero element in $R$ is a unit in $R$. First we note that $R$ is reduced and hence abelian. Let $0 \neq x \in R$ Clearly $x = ur$ where $r = u^{-1}x$ in $R$. Suppose $r \notin U(R) \Rightarrow r \in Id(R) \Rightarrow x(1 - r) = 0 \Rightarrow x \notin U(R) \Rightarrow x \in Id(R)$. Now $x^2 = x \Rightarrow (ur)^2 = ur \Rightarrow u^2r = ur \Rightarrow ur = r \Rightarrow (1 - u)r = 0 \Rightarrow 1 - u \in Id(R) \Rightarrow u \in Id(R)$ which is a contradiction $\Rightarrow r \in U(R)$ and hence $x \in U(R)$. Converse is obvious.

Let $R$ be a commutative ring. Let $M$ be an $R$-module. Then the idealization of $R$ and $M$ is the ring $R(M)$ with underlying set
$R \times M$ under coordinatewise addition and multiplication given by $(r, m) \times (r', m') = (rr', rm' + r'm)$ for all $r, r' \in R$ and $m, m' \in M$. It is clear that $R(M)$ is a commutative ring with identity $(1,0)$.

Before proving the next result we note that:

i) $(r, m) \in Id(R(M))$ if and only if $r \in Id(R)$ and $m = 0$.

For, suppose $(r, m)^2 = (r, m) \Rightarrow (r^2, 2rm) = (r, m) \Rightarrow r^2 = r$ and $2rm = m \Rightarrow 2rm = rm \Rightarrow rm = 0 \Rightarrow m = 0$. Hence $r \in Id(R)$ and $m = 0$. Converse part is very clear.

ii) $(r, m) \in U(R(M))$ if and only if $r \in U(R)$

For, suppose $(r, m) \in U(R(M)) \Rightarrow$ there exists $(s, n) \in R(M)$ such that $(r, m) \ast (s, n) = (1, 0) \Rightarrow rs = 1 \Rightarrow r \in U(R)$. Conversely, suppose $r \in U(R)$ then for any $m \in M$ we have $(r, m) \ast (r^{-1}, -r^{-2}m) = (1, 0)$. Hence $(r, m) \in U(R(M))$.

**Proposition 12:** If $R$ is a commutative ring, then the following holds:

1) $R$ is an exchange ring if and only if so is $R(M)$.

2) $R$ has stable range one if and only if $R(M)$ has stable range 1.

**Proof:**

1) Only if: Let $(a, m) \in R(M) \Rightarrow$ (by hypothesis) there exists $e \in Id(R)$ such that $e = ar$ and $1 - e = (1 - a)s$ for some $r, s \in R$. Then we have (by (i) above) $(e, 0) \in Id(R(M))$ and $(e, 0) = (a, m) * (e, 0) = (a, m) * (a, m) * (e, 0) = (e, 0)$ and $(1 - e, 0) = ((1 - a)s(1 - e), (1 - a)s^2(1 - e)m - s(1 - e)m) = (1 - a, -m) * (s(1 - e), s^2(1 - e)m)$. Hence $R(M)$ is an exchange ring.

If: Let $a \in R \Rightarrow$ (by (i) above and hypothesis) there exists $e \in Id(R)$ such that $(e, 0) = (a, 0) * (b, m)$ and $(1 - e, 0) = (1 - a, 0) * (c, m')$ for some $b, c \in R$ and $m, m' \in M \Rightarrow e = ab$ and $1 - e = (1 - a)c$. Hence $R$ is an exchange ring.

2) Only if: Let $(a, m), (r, s), (b, n) \in R(M)$ such that $(a, m) * (r, s) + (b, n) = (1, 0) \Rightarrow ar + b = 1$ Since $R$ has stable range one, there exists $y \in R$ such that $a + by = u$ for some $u \in U(R) \Rightarrow (a, m) + (b, n) * (y, 0) = (a, m) + (by, ny) = (a + by, m + ny) = (u, m + yn)$ is a unit in $R(M)$ (by (ii) above). Therefore $R(M)$ has stable range one.

If: Let $a, x, b \in R$ be such that $ax + b = 1$ Then $(a, 0) * (x, 0) + (b, 0) = (1, 0)$ in $R(M)$ \Rightarrow (by hypothesis) there exists $(y, z) \in R(M)$ such that $(a, 0) + (b, 0) * (y, z) = (u, m)$ where $(u, m) \in U(R(M)) \Rightarrow$ (by (ii) above) $a + by = u \in U(R)$. Hence $R$ has stable range one.

**References**

1) D.D. Anderson and V.P. Camillo, Commutative rings whose elements are a sum of a unit and idempotent, Comm. Algebra, 30(7), 3327-3336 (2002)


3) V.P. Camillo and H.P. Yu, Stable range one for rings with many idempo-

Received: June, 2010