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Fractal Like Figurates

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Abstract

This paper presents a number of sequences based on integers arranged in a fractal like structure. This approach provides a simple derivation of some well known sequences. In addition, many new integer sequences are obtained.

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1. Introduction

Many well-known sequences can be derived using geometric structures [1],[2],[3],[4]. For example, the number of elements in the square pyramid is

$$s_n = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2 = \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1), \qquad (1)$$

where n is the height of the pyramid. Starting from $n = 1$, we obtain

$$1, 5, 14, 30, 55, \ldots , \qquad (2)$$

which is sequence A000330 in the Encyclopedia of Integer Sequences maintained by Sloane [5], and appropriately called the square pyramidal numbers.

A fractal is a geometric figure that repeats under various levels of magnification. There are many well known fractal shapes such as the Koch snowflake, Sierpinski gasket, and Cantor set [6]. Motivated by these structures, in the next section fractal-like shapes are constructed from squares and rectangles. Sequences based on these shapes are then derived.

2. Fractal-Like Shapes and Integer Sequences

We begin with an example. Consider a square with sides of length 5. Next to this is appended a square with sides of length 4, giving a shape with a side of length 9. Next to the last square is appended a square with sides of length 3. This process is repeated until the square has sides of length 1. This structure is shown in Fig. 1. The area of this figure is

$$5^2 + 4^2 + 3^2 + 2^2 = 54$$

If we start with a square of side length 4, the area is

$$4^2 + 3^2 + 2^2 + 1^2 = 30$$

and in general, if one starts from a square with side length p, the area is

$$p^2 + (p-1)^2 + (p-2)^2 + (p-3)^3 = 4p^2 - 12p + 14 \tag{3}$$

and the resulting integer sequence is

$$30, 54, 86, 126, 174, 230, \ldots$$

This is sequence A027575 in the On-Line Encyclopedia of Integer Sequences [5]. One could also start from a single dot and progress outward in ever increasing squares. The sequence then becomes

$$1, 5, 14, 30, 54, \ldots$$

Substituting $p = 3$ in (3) gives the same value because in this case $p - 3 = 0$, however the results for $p = 1$ and 2 differ. Considering the perimeters, one obtains

$$2p + 2(p-2) + 2(p-1) + 2(p-3) + \ldots + 2(p-5) = 16p - 40$$

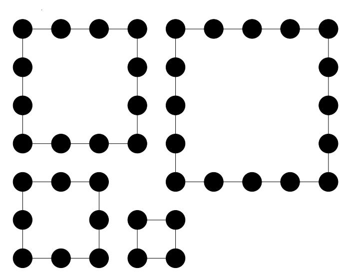

Figure 1: The square construction beginning with side lengths of 5.

This gives the sequence (starting from $p = 5$)

$$40, 56, 72, 88, 104, 120, \ldots$$

which is A051062 in [5].

Another means of deriving (3) is to subtract from the area of the rectangle obtained from the four squares. In Fig. 1, the rectangles has area 9×7, and from this must be subtracted the middle point and the rectangle in the lower right of area 4×2. This gives a final area of $63 - 1 - 8 = 54$, as expected. In general, the area of the large rectangle is $(p + p - 1) \times (p - 1 + p - 2)$ and the are of the lower right rectangle is $4(p - 3)$. Thus the area is

$$(2p - 1) \times (2p - 3) - 4(p - 3) - 1 = 4p^2 - 8p + 3 - 4p + 12 - 1 = 4p^2 - 12p + 14$$

as required.

One can consider shapes other than squares. For example, using rectangles with a side difference of one gives the shape in Fig. 2 for $p = 5$. The area of this shape is

$$6 \times 5 + 5 \times 4 + 4 \times 3 + 3 \times 2 = 68$$

and in general, starting with a rectangle with sides of length p and $p - 1$, we have an area of

$$p(p - 1) + (p - 1)(p - 2) + (p - 2)(p - 3) + (p - 3)(p - 4) = 4p^2 - 16p + 20$$

giving

$$40, 68, 104, 148, 200, \ldots$$

which is sequence A128445 in [5]. The corresponding perimeter sequence is

$$2p + 2(p - 3) + 2(p - 1) + 2(p - 4) + \ldots + 2(p - 6) = 16p - 48$$

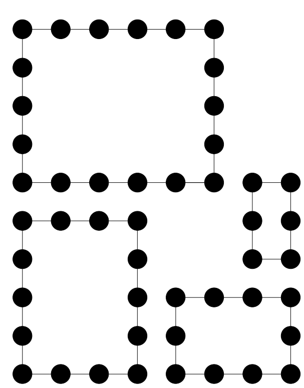

Figure 2: The rectangular construction beginning with side lengths that differ by 1.

In this case, one must start from $p = 6$, giving

$$48, 64, 80, 96, 112, \ldots$$

Note that the sequence for $p = 5$ gives 32, which is correct since $2(p-6)$ gives 2, which is the length of the smallest rectangle. This is sequence A008598 in [5] (which is just the multiples of 16).

Considering rectangles that have sides of lengths p and $p-2$, we obtain an area of

$$p(p-2) + (p-1)(p-3) + (p-2)(p-4) + (p-3)(p-5) = 4p^2 - 20p + 26$$

which gives the sequence

$$50, 82, 122, 170, 226, \ldots$$

It is interesting that this is sequence A069894, twice the centered square numbers [5]. To show this, consider the four terms in the sum

$$\begin{array}{c} p(p-2) \\ (p-1)(p-3) \\ (p-2)(p-4) \\ (p-3)(p-5) \end{array}$$

Adding the first and third, and second and fourth lines gives

$$\begin{array}{rcl} (p-2)(2p-4) & = & 2(p-2)^2 \\ (p-3)(2p-6) & = & 2(p-3)^2 \end{array}$$

Thus the area of the rectangles equals

$$2\left((p-2)^2 + (p-3)^2\right)$$

and the formula for the nth centered square number is (A001844 [5])

$$n^2 + (n-1)^2$$

which proves the result. An illustration of this is given in Fig. 3.

In general, if the difference in side lengths is d, the resulting area is given by

$$\begin{aligned} p(p-d) + (p-1)(p-d-1) + (p-2)(p-d-2) + (p-3)(p-d-3) \\ = 4p^2 - 4pd - 12p + 6d + 14 \end{aligned} \tag{4}$$

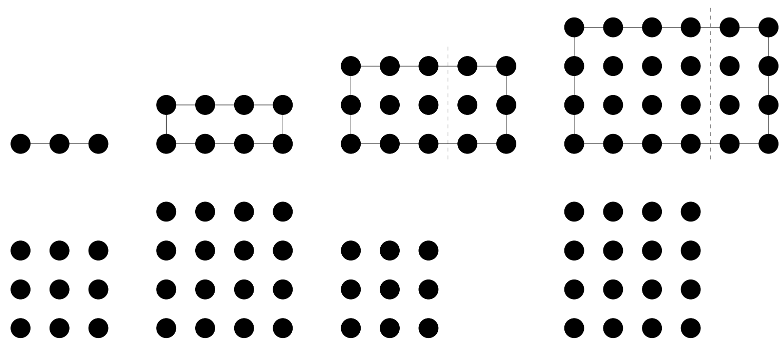

Figure 3: The rectangular construction beginning with side lengths that differ by 2 divided to produce four squares.

for $d = 3$ to 10 we obtain the sequences

$$\begin{array}{rrrrrl} 60, & 96, & 140, & 192, & 252, & \ldots \\ 70, & 110, & 158, & 214, & 278, & \ldots \\ 80, & 124, & 176, & 236, & 304, & \ldots \\ 90, & 138, & 194, & 258, & 330, & \ldots \\ 100, & 152, & 212, & 280, & 356, & \ldots \\ 110, & 166, & 230, & 302, & 382, & \ldots \\ 120, & 180, & 248, & 324, & 408, & \ldots \\ 130, & 194, & 266, & 346, & 434, & \ldots \end{array}$$

where the first element corresponds to $p = d + 4$. The first if these is sequence A134582 [5], while the others are new. One can also derive (4) using subtraction from the rectangle formed from the outer perimeter. The area of this rectangle is

$$(2p - 1 - d) \times (2p - 3 - d) = 4p^2 - 8p - 4pd + 3 + 4d + d^2$$

The centre portion has area $(d + 1)^2$, and the lower right rectangle has area $4(p - 3 - d)$. Thus the area of the four rectangles is

$$4p^2 - 8p - 4pd + 3 + 4d + d^2 - (d + 1)^2 - 4(p - 3 - d) = 4p^2 - 12p - 4pd + 14 + 6d$$

as required.

The corresponding perimeter sequences are given by

$$2(2p - d - 2) + 2(2p - d - 4) + 2(2p - d - 6) + 2(2p - d - 8) = 16p - 8d - 40$$

which for $d = 2$ to 10 gives

$$\begin{array}{cccccc}
56, & 72, & 88, & 104, & 120, & \ldots \\
64, & 80, & 96, & 112, & 128, & \ldots \\
72, & 88, & 104, & 120, & 136, & \ldots \\
80, & 96, & 112, & 128, & 144, & \ldots \\
88, & 104, & 120, & 136, & 152, & \ldots \\
96, & 112, & 128, & 144, & 160, & \ldots \\
104, & 120, & 136, & 152, & 168, & \ldots \\
112, & 128, & 144, & 160, & 176, & \ldots \\
120, & 136, & 152, & 168, & 184, & \ldots
\end{array}$$

where the first element corresponds to $p = d+5$. These are just shifted versions of sequences A051062 and A008598 [5].

One can also construct fractal-like shapes using triangles. The number of points in an equilateral triangle with side p is

$$\frac{p(p+1)}{2}$$

The sequence of triangles gives the well-known triangular numbers

$$1, 3, 6, 10, 15, 21, \ldots$$

which is sequence A000217 [5] Arranged like the squares and rectangles with edges adjacent, we obtain triangles in a spiral shape. As with the rectangular construction, the larger triangles overlap the smaller ones, but in this case starting with the sixth triangle. Thus the area of the triangle spiral is given by

$$\sum_{p-5}^{p} \frac{n(n+1)}{2} = 3p^2 - 12p + 20$$

The corresponding values are

$$56, 83, 116, 155, 200, 251, 308, \ldots$$

This sequence is new. The number of points on the perimeter of an equilateral triangle is $p + p - 1 + p - 2$, thus the total perimeter is

$$\sum_{p-5}^{p} n + n - 1 + n - 2 = 18p - 63$$

The corresponding sequence values are

$$45, 63, 81, 99, 117, 135, 153, 171, \ldots$$

which is also new. It is interesting that the first seven elements of this sequence occur in A031088 and A079297 [5].

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