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# A Note on Generalized Derivations of Prime Rings

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**Abstract**

In the present paper we prove the following result; Let $R$ be a non-commutative prime ring, $I$ an ideal of $R$, $(F, d)$ a generalized derivation of $R$ and $a \in R$. If $F([x, a]) = 0$ or $[F(x), a] = 0$ for all $x \in I$, then, $d(x) = \lambda[x, a]$ for all $x \in I$ or $a \in Z$.

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## 1 Introduction

Throughout $R$ will be a ring with center $Z$. Let $x, y \in R$. The commutator $xy - yx$ will be denoted by $[x, y]$. Recall that a ring is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $\partial : R \to R$ is called a derivation if $\partial(xy) = \partial(x)y + x\partial(y)$ holds for all $x, y \in R$. The study of commutativity of prime rings with derivation was initiated by Posner [6]. During the past few decades, there has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of $R$. Recently, in [2], Bresar defined the following notation. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$.

Throughout in this paper, $R$ will be a prime ring with Martindale ring of quotients $Q_r(R)$, extended centroid $C$ and central closure $R_C = RC$ (see [5, sec. 10], for detail). In [3], B. Havala studied generalized derivation can be uniquely extended to a generalized derivation of $Q_r(R)$. In [1], Albas, E. and Argac, N. showed that $R$ is a non-commutative ring and $d$ a generalized derivation determined by a derivation $\alpha$ of $R$, $a \in R$ and for all $x \in R$, $[a, d(x)] = 0$ then either $a \in C$ or there exist $\lambda, \eta \in C$ such that $d(x) = \eta x + \lambda(ax + xa)$, for all $x \in R$. In this paper, our aim is to extend this result to a non-zero ideal $I$ of $R$. In particular, our research can be viewed as a new more elementary approach.

## 2 Main Results

In the following, we assume that $R$ is a prime ring. We denote a generalized derivation $F : R \to R$ determined by a derivation $d$ of $R$ by $(F, d)$.

**Lemma 2.1** *Let $R$ be a non-commutative ring, $I$ an ideal of $R$ and $(F, d)$ a generalized derivation of $R$ and $a \in R$. If $a \notin Z$ and $[F(x), a] = 0$ for all $x \in I$, then $F([x, a]) = 0$ for all $x \in I$.*
***Proof.*** *We replace $x$ by $xr$, $r \in R$ in the defining equation*

$$[F(x), a] = 0 \text{ for all } x \in I \tag{1}$$

*to obtain,*

$$\begin{aligned} 0 &= [F(xr), a] = [F(x)r + xd(r), a] \\ &= [F(x)r, a] + [xd(r), a] \\ &= F(x)[r, a] + [F(x), a]r + x[d(r), a] + [x, a]d(r) \end{aligned}$$

*for all $x \in I$, $r \in R$ which implies that*

$$F(x)[r, a] + x[d(r), a] + [x, a]d(r) = 0 \tag{2}$$

*for all $x \in I$, $r \in R$. In (2), replace $x$ by $xy, y \in I$ and use (2) , we obtain*

$$\begin{aligned} 0 &= F(xy)[r, a] + xy[d(r), a] + x[y, a]d(r) + [x, a]yd(r) \\ &= F(x)y[r, a] + xd(y)[r, a] + xy[d(r), a] + x[y, a]d(r) + [x, a]yd(r) \\ &= F(x)y[r, a] + xd(y)[r, a] + x(y[d(r), a] + [y, a]d(r)) + [x, a]yd(r) \\ &= F(x)y[r, a] + xd(y)[r, a] - xF(y)[r, a] + [x, a]yd(r) \\ &= (F(x)y + xd(y) - xF(y))[r, a] + [x, a]yd(r) \end{aligned}$$

*so we get*

$$(F(x)y + xd(y) - xF(y))[r, a] + [x, a]yd(r) = 0, \forall x, y \in I, r \in R \tag{3}$$

*Replace $r$ by $a$ in (3), we have*

$$[x, a]Id(a) = 0, \forall x \in I$$

*Since* $a \notin Z$ *and the primeness of* $I$*, yields*

$$d(a) = 0$$

*If we substitute* $sx$, $s \in R$ *for* $x$ *in* (3), *then we get*

$$\begin{aligned}
0 &= (F(sx)y + sxd(y) - sxF(y))[r,a] + [sx,a]yd(r) \\
&= ((F(s)x + sd(x))y + sxd(y) - sxF(y))[r,a] + s[x,a]yd(r) + [s,a]xyd(r) \\
&= (F(s)xy + sd(x)y + sxd(y) - sxF(y))[r,a] + s[x,a]yd(r) + [s,a]xyd(r) \\
&= F(s)xy[r,a] + sd(x)y[r,a] + sxd(y)[r,a] - sxF(y)[r,a] \\
&+ s[x,a]yd(r) + [s,a]xyd(r) \\
&= (F(s)xy + sd(x)y)[r,a] + s\left((xd(y) - xF(y))[r,a] + [x,a]yd(r)\right) \\
&+ [s,a]xyd(r) \\
&= (F(s)xy + sd(x)y)[r,a] + s(-F(x)y[r,a]) + [s,a]xyd(r) \\
&= (F(s)xy + sd(x)y - sF(x)y)[r,a] + [s,a]xyd(r)
\end{aligned}$$

*and so*

$$(F(s)x + sd(x) - sF(x))y[r,a] + [s,a]xyd(r) = 0 \tag{4}$$

*In (4) replacing* $s$ *by* $a$*,*

$$(F(a)x + ad(x) - aF(x))y[r,a] = 0 \ \forall x, y \in I \tag{5}$$

*Using* $a \notin Z$ *and the primeness of* $I$*, we obtain*

$$F(a)x + ad(x) - aF(x) = 0.$$

*Then we have*

$$F(ax) = aF(x), \ \forall x \in I \tag{6}$$

*On the other hand, by employing* $d(a) = 0$*, we see that the relation*

$$F(xa) = F(x)a + xd(a) = F(x)a$$

*is reduced to*

$$F(xa) = F(x)a, \ \forall x \in I \tag{7}$$

*Combining* (6) *and* (7), *we arrive at*

$$F([x,a]) = F(xa) - F(ax) = F(x)a - aF(x)$$

*By using the hypothesis, we have*

$$F([x,a]) = [F(x),a] = 0, \ \forall x \in I$$

*This completes the proof.* ■

**Lemma 2.2** *Let $R$ be a non-commutative prime ring, $I$ an ideal of $R$, $(F,d)$ a generalized derivation of $R$ and $a \in R$. If $a \notin Z$ and $F\ ([x,a]) = 0$ for all $x \in I$, then $[F(x),a] = 0$ for all $x \in I$.*

***Proof.*** *we replace $x$ by $xa$ in the defining equation $F([x,a]) = 0$ to obtain*

$$\begin{aligned} 0 &= F([xa,a]) = F([x,a]a) \\ &= F([x,a])a + [x,a]d(a) \end{aligned}$$

*and so*

$$[x,a]d(a) = 0, \text{ for all } x \in I \tag{8}$$

*Taking $xy$, $y \in I$ instead of $x$ in (8),* $0 = [xy,a]d(a) = x[y,a]d(a) + [x,a]yd(a)$ *and using (8) we obtain*

$$[x,a]Id(a) = 0, \text{ for all } x \in I \tag{9}$$

*By the primeness of $I$ and $a \notin Z$,* (9) *implies that*

$$d(a) = 0$$

*Now we replace $x$ by $xy$, $y \in I$ in the defining equation $F([x,a]) = 0$ to obtain*

$$\begin{aligned} 0 &= F([xy,a]) = F(x[y,a] + [x,a]y) \\ &= F([x,a]y) + F(x[y,a]) \\ &= F([x,a])y + [x,a]d(y) + F(x)[y,a] + xd([y,a]) \\ &= [x,a]d(y) + F(x)[y,a] + x([d(y),a] + [y,d(a)]) \end{aligned}$$

*Since $d(a) = 0$, we have*

$$F(x)[y,a] + [x,a]d(y) + x[d(y),a] = 0, \forall x,y \in I \tag{10}$$

*Substitute $yz$, $z \in I$ instead of $y$ in* (10) *and use* (10)*, we arrive at*

$$\begin{aligned} 0 &= F(x)[yz,a] + [x,a]d(yz) + x[d(yz),a] \\ &= F(x)y[z,a] + F(x)[y,a]z + [x,a]d(y)z + [x,a]yd(z) + x[d(y)z,a] + x[yd(z),a] \\ &= F(x)y[z,a] + (F(x)[y,a] + [x,a]d(y))z + [x,a]yd(z) + xd(y)[z,a] \\ &+ x[d(y),a]z + xy[d(z),a] + x[y,a]d(z) \\ &= F(x)y[z,a] + (F(x)[y,a] + [x,a]d(y) + x[d(y),a])z + [x,a]yd(z) \\ &+ xd(y)[z,a] + xy[d(z),a] + x[y,a]d(z) \\ &= F(x)y[z,a] + [x,a]yd(z) + xd(y)[z,a] + xy[d(z),a] + x[y,a]d(z) \\ &= (F(x)y + xd(y))[z,a] + [x,a]yd(z) + x(y[d(z),a] + [y,a]d(z)) \\ &= (F(x)y + xd(y))[z,a] + [x,a]yd(z) - xF(y)[z,a] \end{aligned}$$

*and so*

$$(F(x)y + xd(y) - xF(y))[z,a] + [x,a]yd(z) = 0, \forall x,y,z \in I \tag{11}$$

*Replace* $x$ *by* $ax$ *in* (11) *and use* (11), *it yields*

$$\begin{aligned} 0 &= (F(ax)y + axd(y) - axF(y))[z,a] + a[x,a]yd(z) \\ &= F(ax)y[z,a] + a(xd(y)[z,a] - xF(y)[z,a] + [x,a]yd(z)) \\ &= F(ax)y[z,a] - aF(x)y[z,a] \end{aligned}$$

*Hence we get*

$$(F(ax) - aF(x))y[z,a] = 0, \forall x,y,z \in I \tag{12}$$

*Since* $a \notin Z$ *and the primeness of I, we have*

$$F(ax) = aF(x), \forall x \in I \tag{13}$$

*On the other hand, since* $d(a) = 0$,

$$F(xa) = F(x)a + xd(a) = F(x)a \tag{14}$$

*Combining* (13) *and* (14) *we arrive at*

$$\begin{aligned} [F(x),a] &= F(x)a - aF(x) \\ &= F(xa) - F(ax) \\ &= F([x,a]) = 0 \end{aligned}$$

*and so*

$$[F(x),a] = 0, \forall x \in I$$

*Thus. the proof is complete.* ■

The following theorem is motivated by [1, Theorem 3.1 and Corollary 3.6].

**Theorem 2.3** *Let* $R$ *be a non-commutative prime ring,* $I$ *an ideal of* $R$, $(F,d)$ *a generalized derivation of* $R$ *and* $a \in R$. *If* $a \notin Z$ *and* $F([x,a]) = 0$ *or* $[F(x),a] = 0$ *for all* $x \in I$, *then* $d(x) = \lambda[x,a]$, *for all* $x \in I$.

***Proof.*** *Since* $a \notin Z$ *and* $[F(x),a] = 0$ *for all* $x \in I$, *then by Lemma 2.1 we have*

$$F([x,a]) = 0 \text{ and } d(a) = 0$$

*By the proof of the Lemma 2.1, we have the equation* (3); *in the equation* (3), *replace* $y$ *by* $[a,y]$ *then we get*

$$\begin{aligned} 0 &= (F(x)[a,y] + xd([a,y]) - xF([a,y]))[r,a] + [x,a][a,y]d(r) \\ &= (F(x)[a,y] + x[a,d(y)][r,a] + [x,a][a,y]d(r) \\ &= -(F(x)[y,a] + x[d(y),a])[r,a] + [x,a][a,y]d(r) \end{aligned}$$

*In the above equation, using the* (10) *equation* $[a,x]d(y) = F(x)[y,a]+x[d(y),a]$ *in the proof of the Lemma 2.1, we obtain*

$$[a,x](d(y)[r,a] - [y,a]d(r)) = 0$$

*Define* $h : R \to R$, $h(x) = [a,x]$, *then the above equation yields* $h(x)(d(y)[r,a]-[y,a]d(r)) = 0$. *Since* $a \notin Z$, *by [6, Lemma 1] we get*

$$d(y)[r,a] = [y,a]d(r), \forall y \in I, r \in R \tag{15}$$

*Replace* $r$ *by* $rs$, $s \in R$ *in* (15) *and use* (15), *we obtain*

$$d(y)r[s,a] = [y,a]rd(s), \forall r,s \in R, y \in I \tag{16}$$

*Substitute* $yz$, $z \in R$ *instead of* $y$ *in* (16) *and use* (16) *it gives us*

$$d(z)r[s,a] = [z,a]rd(s) \tag{17}$$

*Now, define* $g : R \to R$, $g(x) = [x,a]$, *then from* (17) *we have*

$$d(z)rg(s) = g(z)rd(s), \forall r,s,z \in R$$

*Since* $g \neq 0$, *by [4, Lemma 1.3.2] for some* $\lambda \in C$,

$$d(x) = \lambda[x,a]$$

*Thus, the the proof is complete.* ■

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