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Scheme Theory for Groups and Lie Algebras

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Abstract

Many authors have recently studied objects defined by algebraic equations in groups and Lie algebras. On this purpose, they have develop a non commutative classical algebraic geometry. In this paper, we study a theory of schemes of for abstract groups and Lie algebras.

Introduction.

Let $L = \mathcal{F} \bigcup \mathcal{P} \bigcup \mathcal{C}$ be a first order language, $\mathcal{F}$ is the set of symbols of operations of arities n_F , $\mathcal{P}$ is the set of symbols F of predicates, and $\mathcal{C}$ the set of constants. Examples of such languages are: the language of groups, the operations are the multiplication, the inversion, and the neutral is a constant. The language of Lie algebras, the operations here are the addition, the substraction, the Lie bracket, the constants are $0, 1$.

An $\mathcal{L}$-structure is defined by a set M, and for each symbol F in $\mathcal{F}$, a function $F^M : M^{n_F} \rightarrow M$, where n_F is the arity of F, to each predicate is assigned a relation, a constant c_M is assigned to each element c in $\mathcal{C}$; if $\mathcal{L}$ is the language of groups, a $\mathcal{L}$-structure is a group. If $\mathcal{L}$ is the language of Lie algebras, a $\mathcal{L}$-structure is a Lie algebra.

Let $X = \{x_1, .., x_n\}$ a set of variables, the free $\mathcal{L}$-structure $T_{\mathcal{L}}[X]$ is a $\mathcal{L}$-structure universal in the sense that for any $\mathcal{L}$-structure M, any morphism of sets $X \rightarrow M$, extends uniquely to a morphism of $\mathcal{L}$-structures $T_{\mathcal{L}}(X) \rightarrow M$. Example, if $\mathcal{L}$ is the language of groups, $T_{\mathcal{L}}(X)$ is the free group generated by X. The formulas or equation on $T_{\mathcal{L}}(X)$, are defined recursively with elements of $T_{\mathcal{L}}(X)$ and logic operations. If $\mathcal{L}$ is the language of groups, we can always write a set of formulas $(S = 1)$, $S \subset T_{\mathcal{L}}(X)$, if $\mathcal{L}$ is the language of Lie algebras, we can write $(S = 0)$.

In the sequel, we suppose that the language used defined structures which arise in algebra such as: groups, Lie algebras, ring,...

Let M be an $\mathcal{L}$-structure. A M-solution of an equation $(S = 1)$ is a morphism $f : T_{\mathcal{L}}(X) \rightarrow M$ such that $f(S) = 1$. The set of solutions of the equation $(f = 1)$ is denoted $V(S)$. These sets generate the closed subsets of a topology called the Zariski topology.

The algebraic geometry on groups or (Lie) algebras is the study of the geometry of closed subsets for the Zariski topology above. The motivation is to provide a geometric interpretation of subsets of groups and Lie algebras defined by algebraic equations. Like the commutator of a finite subset of a group. The main results of this theory have been stated for groups by Baumslag, Myasnikov, and Remeslennikov [2], and for Lie algebras by Daniyarova, Myasnikov, Remeslennikov [3], Kazachkov [7]. The previous authors, focus on the coordinates systems, the purpose of this paper is to develop a free coordinates approach inspired by the scheme theory defined by Grothendieck.

1 Algebraic geometry of groups.

Firstly, we are going to study the algebraic geometry of groups. Let C be the category of groups, and G an object of C. We denote by $G \mid C$ the comma category whose objects are morphisms $\phi_H : G \rightarrow H$, such an object will usually be denoted (H, ϕ_H). Recall that a morphism $f : (H, \phi_H) \rightarrow (H', \phi_{H'})$ is a morphism of groups $f : H \rightarrow H'$ such that $f \circ \phi_H = \phi_{H'}$. We will be mainly concerned in the full subcategory $C(G)$ of $G \mid C$ whose objects are the objects (H, ϕ_H) such that ϕ_H is injective.

Definition 1.

Let x be an element of the object (H, ϕ_H) of $C(G)$, we denote by $G(x)$, the subgroup of H generated by $\{gxg^{-1}, g \in G\}$. The element x is invertible if and only if $G(x) \cap \phi_H(G) \neq 1$. A non trivial element x of H is a divisor of zero if and only if there exists a non trivial element y of H such that the group of commutators $[G(x), G(y)]$ of elements of $G(x)$ and $G(y)$ is trivial.

A G-domain is an object (H, ϕ_H) of $C(G)$ which does not have zero divisors.

A normal subgroup P of H is a prime ideal if and only if:

- $P \cap \phi_H(G) = 1$
- H/P a G-domain.

This second condition is equivalent to saying that for every element $x, y \in H$, such that $[G(x), G(y)] \subset P, x \in P$ or $y \in P$.

Let I be a normal subgroup of H, we denote by $V(I)$ the set of prime normal subgroups which contains I. We denote by $Spec(H)$ the set of prime normal subgroups of H. We call it an affine G-scheme.

Remarks.

If $I \cap G$ is different of $\{1\}$, then $V(I)$ is empty.

If x is a divisor of zero, for each G-automorphism h, $h(x)$ is a divisor of zero.

Proposition 1.

Let I and J be two normal subgroups of H, $V([I,J]) = V(I) \bigcup V(J)$.

Let I be the normal subgroup of H generated by the family of normal subgroups of H $(I_a)_{a\in A}$, we have $V(I) = \bigcap_{a\in A} V(I_a)$.

Proof.

Let P be a prime subgroup which is an element of $V(I) \bigcup V(J)$, P contains I or J, this implies that P contains $[I,J]$. Thus $V(I) \bigcup V(J) \subset V([I,J])$.

Let P be an element of $V([I,J])$, suppose that P does not contain I and J. There exists elements $x \in I, y \in J$ such that x, y are not in P. The group $G(x) \subset I$, and $G(y) \subset J$, thus the subgroup $[G(x), G(y)]$ of H is contained in $[I,J]$. This implies that x is in P or y is in P a fact which is a contradiction.

Let I be the subgroup of H generated by the family of normal subgroups $(I_a)_{a\in A}$. Consider a prime P which contains I. This implies that P contains I_a for every element $a \in A$. We deduce that $V(I) \subset \bigcap_{a\in A} V(I_a)$. Let P be a prime in $\bigcap_{a\in A} V(I_a)$, P contains $I_a, a \in A$, this implies that P contains I.

The previous proposition shows that there exists a topology on $Spec(H)$ for which the closed subsets can be written $V(I)$, where I is a normal subgroup of H.

Remark that if G is commutative, $Spec(H)$ is always empty. Suppose that $Spec(H)$ is not empty and consider P be an element of $Spec(H)$, for every element x of G, $[G(x), G(x)] = 1$, this implies that $x \in P$. This is a contradiction with the fact that $P \bigcap G$ is empty.

Localization and structural sheaves.

Let (H, ϕ_H) be an object of $C(G)$, we are going to define three sheaves on $Spec(H)$:

Let U be an open subset of $Spec(H)$, we define $L_{Spec(G)}(U)$ to be the set of applications $f : U \to \prod_{P\in U} H/P$, such that for every $P \in U$, there exists an open subset V of U containing P, an element $f_V \in H$ such that for every $Q \in V, f(Q)$ is the image of f_V by the quotient map $u_Q : H \to H/Q$.

Let H be a group, $Z(H)$ is the algebra of H: it is the Z-ring generated by $\{1_h, h \in H\}$. Its multiplication structure is defined by $1_h 1_{h'} = 1_{hh'}$. We denote by $A_{Spec(H)}(U)$ the space of functions $f : U \rightarrow \prod_{P \in U} Z(H/P)$ such that for every $P \in U$, there exists a neighborhood V of P, an element $f_V \in Z(H)$ such that for every $Q \in V$, we have $f(Q) = p_Q(f_V)$, where $p_Q : Z(H) \rightarrow Z(H/Q)$ is the quotient morphism.

To define the last sheaf, we recall the following result on the localization of modules of associative algebras due to Schonfield:

Theorem [1] page 4.

Let R be an associative ring, and Σ a set of maps between finitely generated projective right R-modules, then there exists an associative R_Σ, and a morphism $\alpha : R \rightarrow R_\Sigma$ which is Σ-inverting; this is equivalent to saying that for every map $f : M \rightarrow M' \in \Sigma, f \otimes Id_{R_\Sigma} : M \otimes_R R_\Sigma \rightarrow M' \otimes_R R_\Sigma$ is an isomorphism. The morphism α is universal in the sense that if $h : R \rightarrow S$ is another R_Σ-invertible morphism, then there exists a morphism $h' : R_\Sigma \rightarrow S$, such that $h = h' \circ \alpha$.

Let P be an element of $Spec(H)$, and $p_P : Z(H) \rightarrow Z(H/P)$ the natural projection. For any subset D of $Spec(H)$, we denote by Σ_D the subset of elements of $Z(H)$ such that for every $f \in \Sigma_D$ and every elements P of $D, p_P(f) \neq 0$. Each element of Σ_D defines a morphism of the right module $Z(H)$ by left multiplication. We denote by $Z(H)_D$ the localization of $Z(H)$ by Σ_D and by α_D the inverting morphism of Σ_D.

Let U be an open subset of $Spec(H)$, we denote by $O_{Spec(H)}(U)$ the set of functions $f : U \rightarrow \prod_{P \in U} Z(H)_P$, such that for every element $P \in U$, there exists a neighborhood V of P in U, an element $f_V \in Z(H)_V$ such that for every element Q in V, $f(Q)$ is the image of f_V by the canonical morphism $Z(H)_V \rightarrow Z(H)_P$ resulting from the universal property of $Z(H)_P$.

Proposition 2.

Let (H, ϕ_H) and $(H', \phi_{H'})$ be elements of $C(G)$, and $f : H \rightarrow H'$ be a morphism, then f induces a continuous map $f^* : Spec(H') \rightarrow Spec(H)$, a morphism of sheaves $f_L : L_{Spec(H)} \rightarrow L_{Spec(H')}$, morphisms of ringed spaces $f_O : (Spec(H'), O_{Spec(H')}) \rightarrow (Spec(H), O_{Spec(H)}), f_A : (Spec(H'), A_{Spec(H')}) \rightarrow (Spec(H), A_{Spec(H)})$.

Proof.

Let P be an element of $Spec(H')$, we set $f^*(P) = f^{-1}(P)$. The morphism $H/f^{-1}(P) \rightarrow H'/P$ is injective, it implies that $f^{-1}(P)$ is a point of $Spec(H)$, since a sub G-group of a G-group which does not have zero divisors, does not have zero divisors.

Let I be a normal subgroup of $H', f^{-1}(V(I)) = V(f^{-1}(I))$. This implies that f^* is continuous.

Let U be an open subset of $Spec(H')$, we define $f_O(U) : O_{Spec(H)}({f^*}^{-1}(U)) \to O_{Spec(H')}((U)$ as follows: Remark that $f^{-1}(\Sigma_U) \subset \Sigma_{{f^*}^{-1}(U)}$. Thus f induces a morphism $Z(H)_{f^{*-1}(U)} \to Z(H')_U$ which is $f_O(U)$. The maps f_L and f_A are defined analogously.

Definition 2.

A group G-scheme is a ringed space (X, O_X), such that every element of X has a neighborhood isomorphic to an affine G-scheme.

Representations and schemes.

Let L be a G-domain, $\{1\}$ is a prime, it is the generic point of $Spec(H)$. Suppose that H is a G-domain, An L-point of $Spec(H)$ is a morphism $i_L : Spec(L) \to Spec(H)$. The morphism i_L is defined by a morphism of G-groups $h_P : H \to L$. We denote by $P = h_P^{-1}(1)$. In particular, if P is an element of $Spec(H)$ such that $H/P = L$, it is an L-point. The purpose of this part is to show that any affine scheme can be realized as a set of representations: There exists a group L, such that every element of $Spec(H)$ is an L-point. On this purpose we show the following:

Lemma 1.

Let $(L_i)_{i\in I}$ be a family of G-domains, the amalgamated sum $\coprod_G L_i, i \in I$ is a G-domain.

Proof.

Since the morphism $\phi_i : G \to L_i$ is injective, the canonical imbedding $i_{L_i} : L_i \to L$ is injective (See Serre Theorem 1 p.9). Let x and y be two elements of L. Suppose that $[G(x), G(y)] = 1$. If there exists an element $i \in I$ such that $x = i_{L_i}(x_i), y = i_{L_i}(y_i)$ we deduce that $[G(x_i), G(y_i)] = 1$ since i_{L_i} is injective. We deduce that $x = 1$ or $y = 1$ since L_i is an H-domain. If such an i does not exist, necessarily $[G(x), G(y)] \neq 1$, since L is the quotient of the free product of $(L_i)_{i\in I}$ by the normal subgroup generated by $i_{L_i}(g)i_{L_j}(g)^{-1}, g \in G$.

Theorem 1.

Let (H, ϕ_H) be an object of $C(G)$. There exists a group L, such that every point of $Spec(H)$ is an L-point.

Proof.

Let L be the amalgamated sum of H/P, $P \in Spec(H)$. Then the previous lemma shows that L is a G-domain. Let $l_P : H/P \to L$, be the canonical imbedding. The composition of the canonical projection $H \to H/P$ with l_P endows P with the structure of an L-point.

Schemes defined by finite groups.

We are going to study here the theory developed above for finite groups. We start by the following result:

Proposition 3.

Let G be a finite group, there exists a group L such that for every (H, ϕ_H) of $C(G)$ where H is finite, the points of $Spec(H)$ are L-points.

Proof.

Let n be an integer, the set of isomorphic classes of G-domain of cardinality inferior to n is finite. This implies that the set of isomorphic classes of finite G-domains is numerable. Let $(L_i)_{i\in I}$ be a set of G-domains such that every G-domain is isomorphic to a L_i. We can suppose that L_i is finite or numerable. We define the amalgamated sum $\coprod_G L_i, i \in I$. If $Spec(H)$ is a G-affine scheme, for every point P of $Spec(H)$, H/P is a finite G-domain. There exists an imbedding $H/P \to L$ whose composition with $H \to H/P$ defines a L-point.

Let G be a group, and $Aut(G)$ the group of automorphisms of G. An inner automorphism i_g of G is defined by an element $g \in G$ such that for every element x of G, $i_g(x) = gxg^{-1}$. We call $In(G)$ the group of inner automorphisms G, and by $Out(G)$ the quotient $Aut(G)/Inn(G)$. For a large class of groups, $Out(G) = 1$ in that situation, we say that G is complete. For example if $G = S_n, n \neq 2, 6$, G is complete.

Proposition 4.

Suppose that H is a finite G-domain where $Out(G) = 1$, then the normalizer of G in H is G.

Proof.

Let $N(G)$ be the normalizer of G in H. Consider the map: $N(G) \to Aut(G), n \to i_n$, the restriction of the inner morphism defined by n to G. If $N(G)$ is not equal to G, then the kernel of i is not trivial. Let x be a non trivial element in the kernel of i, x commutes with G. This implies that $[G(x), G(x)] = 1$. Thus x is a divisor of zero. This is a contradiction.

We give an example of a domain:

Proposition 5.

Let $G = S_n, n > 4$ be the symmetric group, then S_n is an S_n-domain; S_{n+1} endowed with the S_n-structure defined by the canonical imbedding $S_n \to S_{n+1}$ is an S_n-domain.

Remark.

Another topology can be defined on the category of groups as follows: Let G be a group, we say that a normal subgroup P of G is prime ideal if and only if for every normal subgroups I, J of $G, [I, J] \subset P$ if and only if $I \subset P$ or $J \subset P$. We denote by $Spec(G)$ the set of prime ideals of G. Let I be a normal subgroup of G, we denote by $V(I)$ the set of prime ideals of G which contain I. The sets $V(I)$, are the closed subsets of a topology on $Spec(H)$. The problem with this approach is due to the fact that it is not functorial: if H is a finite commutative group, $Spec(H)$ is empty, if G is a simple group, $Spec(G)$ contains only one element. They may exist an imbedding $H \to G$ which does not induce a morphism $Spec(G) \to Spec(H)$.

The affine scheme associated with equations.

The motivation of the study of algebraic geometry for groups is the study of algebraic equations in group theory. Let G be a group, and $X = \{x_1, ..., x_n\}$ a finite set. Consider $H = G * F(x_1, ..., x_n)$, the the free product of G with the free group $F(x_1, ..., x_n)$ generated by X. Many interesting sets in group theory can be expressed with ideals in H. For example, if g is an element of G, we can study the set $E(g)$ of n-uples $(g_1, ..., g_n) \in G^n$ such that $g_i g = g g_i$. This n-uples are defined by the equations $g x_i g^{-1} 1 {x_i}^{-1} = 1, i = 1, .., n$ which generate a normal subgroup $I(g)$ of H. The elements of $E(g)$ can be identified with G-morphisms $H \to G$ whose kernel contains $I(g)$. Such system of equations are studied by Baumslag and his coauthors. To study the algebraic equations on H, we can also study the algebraic scheme defined by its G-structure.

Algebraic geometry of Lie algebras.

We are going firstly to study, the topology on the set of maximal ideals of a Lie algebra Let Lie be the category of Lie algebras and $Lie(\mathcal{S})$ the comma category whose objects are morphisms $\mathcal{S} \to G$. We study here the full subcategory $C(\mathcal{S})$ of $C(\mathcal{S})$ whose objects are injective morphisms $\phi_{\mathcal{G}} : \mathcal{S} \to \mathcal{G}$. We denote this object by $(\mathcal{G}, \phi_{\mathcal{G}})$. Recall that a morphism between $f : (\mathcal{G}, \mathcal{G}) \to (\mathcal{H}, \phi_{\mathcal{H}})$ is a morphism of Lie algebras $f : \mathcal{G} \to \mathcal{H}$ such that $\phi_{\mathcal{H}} = f \circ \phi_{\mathcal{G}}$. For every element $x \in (\mathcal{G}, \phi_{\mathcal{G}})$, we denote by $\mathcal{S}(x)$ the orbit of x by $\mathcal{S}$. The category $Lie(\mathcal{S})$ has limits and colimits, since the category of Lie algebras has limits and colimits. A direct construction of sum in $Lie(\mathcal{S})$ can be done follows: Let

$(\mathcal{G}, \phi_{\mathcal{G}})$ and $(\mathcal{H}, \phi_{\mathcal{H}})$ two objects $C(\mathcal{S})$. The sum of $(\mathcal{G}, \phi_{\mathcal{G}})$ and $(\mathcal{H}, \phi_{\mathcal{H}})$ is the pushout of $\mathcal{G}$ and $\mathcal{H}$ by $\phi_{\mathcal{G}}$ and $\phi_{\mathcal{H}}$. It is the quotient of $\mathcal{G} \oplus \mathcal{H}$ by the ideal generated by $\{\phi_{\mathcal{G}}(s) - \phi_{\mathcal{H}}(s), s \in \mathcal{S}\}$.

The product of $(\mathcal{G}, \phi_{\mathcal{G}})$ and $(\mathcal{H}, \phi_{\mathcal{H}})$ endowed with the diagonal action of $\mathcal{S}$ defines a product in $C(\mathcal{S})$.

Definition 3.

A non zero element $x \in (\mathcal{G} - \mathcal{S})$ is a divisor of zero, if there exists a non zero element $y \in \mathcal{G} - \mathcal{S}$ such that $[\mathcal{S}(x), \mathcal{S}(y)] = 0$.

A prime ideal of $(\mathcal{G}, \phi_{\mathcal{G}})$ is an ideal P of $\mathcal{G}$ such that:

- $P \cap \phi_{\mathcal{G}}(\mathcal{S}) = \{0\}$.

-Let $\phi_P : \mathcal{G} \to \mathcal{G}/P$ be the canonical projection, $(\mathcal{G}/P, \phi_P \circ \phi_{\mathcal{G}})$ does not have a divisor of zero. This second condition is equivalent to saying that for every elements x, y of $\mathcal{G}$ such that $[\mathcal{S}(x), \mathcal{S}(y)] \subset P, x \in P$ or $y \in J$.

For each ideal I of $\mathcal{G}$, we denote by $V(I)$ the set of prime ideals which contain I, and by $Spec(\mathcal{G})$ the set of prime ideals of $\mathcal{G}$.

Proposition 6.

Let $\mathcal{G}$ be a Lie algebra, for every ideals I, J of $\mathcal{G}$, we have: $V([I, J]) = V(I) \bigcup V(J)$. For any family of ideals $(I_a)_{a \in A}$, we have: $V(\oplus_{a \in A} I_a) = \bigcap_{a \in A} V(I_a)$..

Proof.

We firstly show that $V([I, J]) = V(I) \bigcup V(J)$. Let $P \in V(I) \bigcup V(J)$. Since P contains I or J, it contains $[I, J]$. This implies that $V(I) \bigcup V(J) \subset V([I, J])$.

Let P be an element of $V([I, J])$, Suppose that there exists $x \in I, y \in J$ which are not elements of P. Since I and J are ideals, $x \in I$, and $y \in J$, we deduce that $\mathcal{S}(x) \in I, \mathcal{S}(y) \in J$, and $[\mathcal{S}(x), \mathcal{S}(y)] \subset P$. we deduce that $x \in P$ or $y \in P$ since P is a prime ideal.

Now we show that $V(\oplus_{a \in A} I_a) = \bigcap_{a \in A} V(I_a)$. Let P be an element of $V(\oplus_{a \in A} I_a), P$ contains $\oplus_{a \in A} I_a$. This implies that $I_a \subset P$, for every $a \in A$, it results that $P \in V(I_a)$. Thus $P \in \bigcap_{a \in A} V(I_a)$.

Conversely, let P be an element of $\bigcap_{a \in A} V(I_a), I_a \subset P$ for every $a \in A$. This implies that $\bigcap_{a \in A} V(I_a) \subset P$.

Localization and the structural sheaf.

Let $(\mathcal{G}, \phi_{\mathcal{G}})$ be an element of $C(\mathcal{S})$, we denote by $E(\mathcal{G})$ the enveloping of $\mathcal{G}$. Recall that that we can imbed $\mathcal{G}$ in the Lie algebra defined by the commutator bracket of $E(\mathcal{G})$. For every element $P \in Spec(G)$, we denote by $p_P : \mathcal{G} \to \mathcal{G}/P$ the canonical projection, and by $E(p_P) : E(\mathcal{H}) \to E(\mathcal{G}/P)$ the induced map on the enveloping algebras. Let D be a subset of $Spec(\mathcal{G})$, we denote by Σ_D the subset of $E(\mathcal{G})$ such that for every element $P \in D$, and every element $h\Sigma_D, E(p_P)(h) \neq 0$. The elements of Σ_D induces on $E(\mathcal{G})$ morphisms of the right $E(\mathcal{G})$-module $E(\mathcal{G})$ by left multiplications. We denote by $E(\mathcal{G})_D$ the localization of $E(\mathcal{G})$ by Σ_D, and by $\Sigma_D : E(\mathcal{G}) \to E(\mathcal{G})_D$ the inverting morphism.

Let U be an open subset of $Spec(\mathcal{G})$, and $O_{\mathcal{G}}(U)$ the set of maps $f : U \to \prod_{P \in U} E(\mathcal{G})_P$, such that for every $P \in U$, there exists an open subset V containing P, an element $f_V \in E(\mathcal{G})_V$, such that for every element $Q \in V, f(Q)$ is the image of f_V by the canonical morphism $E(\mathcal{G})_V \to E(\mathcal{G})_P$ resulting from the universal property of $E(\mathcal{G})_P$.

Proposition 7.

Let $f : \mathcal{G} \to \mathcal{H}$ be a morphism of $\mathcal{S}$-algebras, For every prime ideal P of $\mathcal{H}$, the inverse image $f^{-1}1(P) = f'(P)$ is also prime, The morphism $f' : Spec(\mathcal{H}) \to Spec(\mathcal{G})$ is continuous.

Proof.

Firstly we show that if P is a prime ideal of $\mathcal{H}$, then $f^{-1}(P)$ is also a prime ideal of $\mathcal{G}$. The morphism f induces an injective map $\bar{f} : \mathcal{G}/f^{-1}(P) \to \mathcal{H}/P$. It results that $\mathcal{G}/f^{-1}(P)$ does not have zero divisors, since $\mathcal{H}/P$ does not have zero divisors.

Let I be an ideal of $\mathcal{H}, f'^{-1}(V(I)) = V(f(I))$. This implies that f is continuous.

Definition 4.

A Lie $\mathcal{S}$-scheme is a topological space X, endowed with a sheaf O_X such that for every element x of X, there exists a neighborhood U_x of x such that $(U_x, O_X(U_x))$ is homeomorphic to an affine $\mathcal{S}$-scheme $Spec(\mathcal{G}, O_{\mathcal{G}})$.

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