Idempotents in Semimedial Semigroups

Fitore Abdullahu

Department of Mathematics
University of Prishtina
Avenue "Agim Ramadani" 58
Prishtinë 10000, Kosova
fmabdullahu@yahoo.com

Abstract

We say that a semigroup $S$ is a I-semimedial semigroup if it is both I-left $(x^2yz = xxyz)$ and I-right $(zyx^2 = zxyx)$ semimedial, where $x, y, z$ belong to $S$ and where $x$ and $y$ are idempotents. In this paper it is presented the connection between I-semimedial semigroups and some special classes of semigroups. It is proved that the class of I-semimedial semigroups, I-medial semigroups and I-distributive semigroups are the same. It is also proved that the left semimedial weakly separative semigroup has no left magnifying element.

Mathematics Subject Classification: 20M10

Keywords: I-semimedial semigroups, I-distributive semigroups

1 Introduction

A semigroup $S$ is called medial semigroup if $xyzu = zxyu$ where $x, y, z, u$ belong to $S$. We say that a semigroup $S$ is a left (right, resp.)
semimedial if it satisfies the identity \( x^2yz = xyxz , \ ( zyx^2 = zyxz , \) resp.). The semigroup \( S \) is a semimedial if it is both left and right semimedial. The class of semimedial semigroups is considered in [1] and [7].

The notion of I-medial semigroups and I-commutative semigroups was introduced in [3]. In [3] Gangnon proved some general results about regular I-medial semigroups.

This paper deals with semigroups which are I-right semimedial and I-left semimedial. First of all we cite some results which will be used in our investigation. Secondly, we present the connection between I-semimedial semigroups, I-distributive semigroups and some special classes of semigroups. It is shown that the I-semimedial semigroups, I-distributive semigroups and I-medial semigroups are the same. Finally, it is shown that the left semimedial weakly separative semigroup has no left magnifying element. For notations and notions not defined here, we refer to [5] and [6].

2 Preliminares

A semigroup \( S \) is called:
- I-left (right, resp.) commutative if it satisfies the identity \( xyz = yzx , \ ( zxy = zyx , \) resp.), where \( x, y \) are idempotents;
- By [3] I-commutative if it satisfies the identity \( xy = yx , \) where \( x, y \) are idempotents;
- By [3] I-medial if it satisfies the identity \( xyzu = xzyu \) where \( y, z \) are idempotents;
- I-left (right, resp.) semimedial if it satisfies the identity \( x^2yz = xyxz , \ ( zyx^2 = zyxz , \) resp.), where \( x, y \) are idempotents;
- I-semimedial if it is both I-left and I-right semimedial;
- I-left (right, resp.) distributive if it satisfies the identity \( xyz = xyxz , \ ( zyx = zxyx ) , \) where \( x, y \) are idempotents;
- I-distributive if it is both I-left and I-right distributive;
- By [2] diagonal if it is satisfy the identities \( x^2 = x, yz = xz ; \)
- By [4] weakly separative if for any \( x, y \in S , \ x^2 = xy = y^2 \) implies \( x = y . \)

It is clear that every commutative (medial, semimedial, distributive, resp.) semigroup is I-commutative (I-medial, I-semimedial, I-distributive, resp.) semigroup.

**Proposition 1** (i) ([3]) Every I-commutative semigroup is I–medial;
(ii) Every I-medial semigroup is I semimedial;
(iii) Every I-semimedial semigroup is I-distributive.

PROOF: It is easy. □

Proposition 2 (a) Every I-left komutative semigroup is I-left semimedial;
(b) Every diagonal semigroup is I-semimedial.

PROOF: It is easy. □

The converse of Proposition 1, 2 is not true, see examples.

Example 3 The semigroup S given by the table

<table>
<thead>
<tr>
<th>*</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
</tbody>
</table>

is I-semimedial semigroup and it is not I-commutative (diagonal, I-left commutative, resp.) semigroup since $b*c \neq c*b$ ($a^2 \neq a, b*c*a \neq c*b*a$, resp.).

Example 4 The semigroup S given by the table

<table>
<thead>
<tr>
<th>*</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>b</td>
<td>b</td>
<td>c</td>
</tr>
</tbody>
</table>

is I-commutative (I-left commutative, I-semimedial, ) semigroup and it is not diagonal since $a^2 \neq a$.

The examples above imply that every I-semimedial semigroup is not I-commutative (diagonal, I-left commutative).

Corollary 5 ([1]) Let $S$ be an idempotent semigroup. Then:
(i) $S$ is left semimedial if and only if $S$ is left distributive;
(ii) $S$ is semimedial if and only if $S$ is a medial.

An element $a$ of semigroup $S$ is called
- completely regular element if exists \( x \in S \) such that \( a = axa, ax = xa \).
- a left magnifying element if exists a proper subset \( M \) of \( S \) such that \( aM = S \).

**Lema 5** ([4]) If \( a \) is a completely regular element of a semigroup \( S \), then \( a \) is not a left magnifying element of \( S \). □

### 3 Main results

**Proposition 6** The following conditions are equivalent for a semigroup \( S \):

(i) \( S \) is I-medial
(ii) \( S \) is I-semimedial
(iii) \( S \) is I-distributive

**PROOF.** It is clear that \((i) \Rightarrow (ii) \Rightarrow (iii)\).

\((iii) \Rightarrow (i)\) Let \( S \) be a I-distributive semigroup. By using I-distributivity

\[ xyzu = x \cdot yzu = x \cdot yzyu = xzyy u = xyz u = xzyu, \]

where \( x, y, z, u \) belong to \( S \) and where \( y, z \) are idempotents.

Then, \( S \) is I-medial semigroup. □

**Lemma 7** I-left semimedial semigroup \( S \) satisfying \( yxy = x \) (where \( x, y \) are idempotents) is I-commutative.

**PROOF.** Let \( S \) be a I-left semimedial semigroup satisfies \( yxy = x \). Then we have

\[ xy = xyxy = xyyx = x \cdot yxy = xx = yxyx = yyxx = yx. \]

**Lemma 8** I-left semimedial semigroup \( S \) satisfying \( xy = y \) (where \( x, y \) are idempotents) is I-left commutative.

**PROOF.** Let \( S \) be a I-left semimedial semigroup satisfies \( xy = y \). Then we have

\[ xyz = xxyz = xyyx = yx. \]

**Lemma 9** Let \( S \) be an idempotent semigroup. Then the following conditions are equivalent:

(i) \( S \) is diagonal semigroup;
(ii) \( S \) is left semimedial satisfies \( xyx = x \).

**PROOF:** It is clear that \((i) \Rightarrow (ii)\). Assume \((ii)\). Then we have

\[ xyz = xyyz = xyyx = xz. \] Hence, \( S \) is diagonal semigroup. □
**Lemma 10** Let $S$ be a left (right resp.) semimedial semigroup and $e$ an idempotent of $S$, then
\[ R_e = \{(x,y) \in S \times S : ex = ey\}, \quad (R_e = \{(x,y) \in S \times S : xe = ye\}, \text{resp.} * \]
is congruence on $S$.

**Proof.** Let $S$ be a left semimedial semigroup. It is clear that $R_e$ is equivalence. Let $x, y, z \in S$ so that $(x, y) \in R_e$. Then
\[ exz = eezx = ezex = ezev = eezy = eyz \]
so $(zx, zy) \in R_e$. It is clear that $R_e$ is right congruence on $S$. □

**Corollary 11** Let $S$ be a left (right) distributive semigroup and $e$ an idempotent of $S$, then $R_e (\ast)$ is congruence on $S$.

**Proof.** Similar to that of Lemma 10. □

**Theorem 12** Left semimedial weakly separative semigroup has no left magnifying element.

**Proof.** Let $S$ be a semimedial weakly separative semigroup. Assume that $S$ has a magnifying element $a$. Then, a proper subset $M$ of $S$ exists: $aM = S$. It is clear $aS = S, a^2S = S \Rightarrow$
\[ \exists x \in S : \quad a = a^2x \]
As $S$ is semimedial semigroup we have
\[ a^2 = a^2xa = aaxa = aaxa = axa^2xa = axaaxa = (axa)^2 \]
From weakly separativity $a = axa, a$ is a regular element.
Since $x \in S = aS, x = az$ for some $z \in S$. Using left semimediality
\[ ax = a^2xx = aaxx = axax = aax = azax = axax = xaax = xa^2x = xa, \quad a \text{ completely a regular element. By Lemma 5, } a \text{ is not a left magnifying element of } S. \quad \square \]

**References**


Received: August, 2010