The Kronecker Product of Symmetric Group Representations Using Schur Functions

N. Ravi Shankar¹, Ch. Suryanarayana², S. Raja Sekhar³, A. Ganapathi Rao⁴

¹Dept. of Applied Mathematics, GIS, GITAM University, Visakhapatnam, India
dravi68@gmail.com
²Dept. of Applied Mathematics, Andhra University, Visakhapatnam, India
³Dept. of Mathematics, JNTU, Kakinada, India
⁴Dept. of BS&H, GMRIT, Rajam, Srikakulam, India

Abstract

The Kronecker product of two Schur functions $s_{\lambda}$ and $s_{\mu}$ is the Frobenius characteristic of the tensor product of the irreducible representations of the symmetric group corresponding to the diagrams $\lambda$ and $\mu$. Taking the kronecker product of $s_{\lambda} \otimes s_{\mu}$ with a third Schur function $s_{\nu}$ gives the so-called Kronecker coefficient $g_{\lambda,\mu,\nu} = \langle s_{\lambda} \otimes s_{\mu}, s_{\nu} \rangle$ which gives the multiplicity of the representation corresponding to $\nu$ in the tensor product. In this paper, we give an algorithm for expanding the Kronecker product $s_{(n-r,\lambda)} \otimes s_{(n-s,\mu)}$, where $\lambda$ and $\mu$ are partitions of $r$ and $s$ respectively for all $n$.

Mathematics Subject Classification: 05E05, 05E10, 17B10, 17B35, 20C30

Keywords: Young tableau, Schur function, Littlewood-Richardson Rule, Kronecker product

1. Introduction

Let $\chi^\lambda$ and $\chi^\mu$ be the irreducible characters of symmetric group $S_n$ indexed by the partitions $\lambda$ and $\mu$ of positive integer $n$. The Kronecker product $\chi^\lambda \chi^\mu$ is the
character that corresponds to the diagonal action of $S_n$ on the tensor product of the irreducible representations indexed by $\lambda$ and $\mu$. Then we have

\[ \chi^\lambda \otimes \chi^\mu = \sum_\nu g_{\lambda, \mu, \nu} \chi^\nu, \]  

(1)

where $\nu$ is a partition of $n$ and $g_{\lambda, \mu, \nu}$ is the multiplicity of $\chi^\nu$ in $\chi^\lambda \otimes \chi^\mu$. By means of the Frobenius map we can define the Kronecker (inner) product on the Schur symmetric functions by

\[ s_\lambda \otimes s_\mu = \sum_\nu g_{\lambda, \mu, \nu} s_\nu. \]  

(2)

In [3-5], derived closed formulas for Kronecker products of Schur functions indexed by two row shapes or hook shapes. Gessel [6] obtained a combinatorial interpretation of zigzag partitions. Dvir [7] has given for any $\lambda$ and $\mu$ a simple and precise description for the maximum length of $\nu$ and the maximum size of $\nu_1$ whenever $g_{\lambda, \mu, \nu}$ is non zero. Stembridge [2] obtained the complete classification of multiplicity-free products of Schur functions, or equivalently, outer products of characters of the symmetric groups. Ballantine and Orellana [8] have given a formula for $g_{(n-k,p),\lambda,\nu}$ in terms of the Littlewood-Richardson coefficients which does not involve cancellations. In [1], the multiplicity-free products of Schur $P$-functions are classified, and then it is applied to the case of projective outer products of spin characters of the double covers of the symmetric group. In this paper, we give an algorithm for expanding the Kronecker product $s_{(n-k,\lambda)} \otimes s_{(n-k,\mu)}$, where $\lambda$ and $\mu$ are partitions of $r$ and $s$ respectively for all $n$.

2. Preliminaries

Let $X = \{ x_1, \ldots, x_k \}$ be a set of variables and let $\Lambda$ be the algebra of symmetric functions in $X$. Bases for this algebra are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, i.e., $\lambda$ is a weakly decreasing sequence of $k$ nonnegative integers $\lambda_i$, called parts. Associated with any partition is an alternant which is the $k \times k$ determinant

\[ a_\lambda = \det(x_i^{\lambda_j}). \]  

(3)

In particular for the partition $\delta = (k-1,k-2,\ldots,0)$ we have Vandermonde’s determinant

\[ a_\delta = \prod_{i<j} (x_i - x_j). \]  

(4)

Schur defined the functions which bear his name as

\[ s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}. \]  

(5)
where addition of partitions is component-wise. It is clear from Eq. (5) that $s_\lambda$ is symmetric homogeneous polynomial of degree $|\lambda| = \sum \lambda_i$.

There is a more combinatorial definition of Schur function. A partition $\lambda$ can be viewed as a Ferrers shape obtained by placing dots or cells in $k$ left justified rows with $\lambda_i$ boxes in row $i$. One obtains a Semistandard Young Tableaux (SSYT), $T$, of shape $\lambda$ by replacing each cell by a positive integer so that rows weakly increase and columns strictly increase. For example, if $\lambda = (4,2,1)$ then its shape and a possible tableau are

$$\lambda = \begin{array}{ccc}
& & \\
& & \\
& & \\
\end{array} \quad T = \begin{array}{ccc}
1 & 1 & 1 \\
2 & 3 \\
4 & \\
\end{array}$$

Each tableau determines a monomial $X^T = \prod_{i \in T} x_i$, e.g., in our example $X^T = x_1^3x_2x_3x_4$. Our second definition of the Schur function is then

$$s_\lambda = \sum_T X^T$$

where the sum is over all SSYT of shape $\lambda$ with entries between 1 and $k$.

The Schur functions can also be written in terms of the other standard bases for $\Lambda$. A monomial symmetric function $m_\lambda$ is the sum of all monomials whose exponent sequences is some permutation of $\lambda$. Also define the Kostka number $K_{\lambda\mu}$ as the number of SSYT $T$ of shape $\lambda$ and content $\mu = (\mu_1, \ldots, \mu_k)$, i.e., $T$ contains $\mu_i$ entries equal to $i$ for $1 \leq i \leq k$. The combinatorial definition of $s_\lambda$ immediately gives the Young’s rule

$$s_\lambda = \sum_{\mu} K_{\lambda\mu} m_{\mu}.$$  \hspace{1cm} (7)

Consider the complete homogeneous symmetric functions $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$ and the elementary symmetric functions $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ where $h_{\lambda_i}$ ($e_{\lambda_i}$) is the sum of all (all square free) monomials of degree $\lambda_i$. Also let $\lambda'$ denote the partition conjugate to $\lambda$ whose parts are the column lengths of $\lambda$’s shape. For the two bases under consideration the $s_\lambda$ can be described as a determinant

$$s_\lambda = \det(h_{\lambda-i+1}),$$ \hspace{1cm} (8)

$$s_\lambda' = \det(e_{\lambda-i+1})$$ \hspace{1cm} (9)
3. A formula for the coefficients in $s_{(n-r,\lambda)} \otimes s_{(n-s,\mu)}$

It is well known that [9] a Schur function can be expanded into a sum of products of elementary symmetric functions.

$$s_{(\mu_1, \mu_2, \ldots, \mu_q)} = s_{[\mu_1]} = s_{[\mu_2] = s_{[\mu_3]} = s_{[\mu_q]}}$$

where $(\mu_1, \mu_2, \ldots, \mu_q)$ is the partition conjugate to the partition $(\lambda_1, \lambda_2, \ldots, \lambda_p)$ and the subscripts $s, t$ represents the corresponding rows and columns of the determinant.

For $s_{(\delta_1, \delta_2)}$, consider $S_{\delta} = \lambda$ to reduce the steps.

Therefore, $$(\delta_1, \delta_2) = \begin{bmatrix} h_{\delta_1} & h_{\delta_1+1} \\ h_{\delta_2-1} & h_{\delta_2} \end{bmatrix} = \begin{bmatrix} (\lambda_1) & (\lambda_1 + 1) \\ (\lambda_2 - 1) & (\lambda_2) \end{bmatrix} = (\lambda_1)(\lambda_2) - (\lambda_1 + 1)(\lambda_2 - 1).$$

Hence,

$$s_{(\delta_1, \delta_2)} = s_{(\lambda_1)} (\delta_2) - (\delta_1 + 1)(\delta_2 - 1)$$

(11)

Using $(\lambda_1, \lambda_2, \lambda_3)$, we obtain

$$s_{(\delta_1, \delta_2, \delta_3)} = s_{(\delta_1, \delta_2, \delta_3)} - [s_{(\delta_1+1, \delta_2)} + s_{(\delta_1, \delta_2+1)}] s_{(\delta_3)} + s_{(\delta_1+1, \delta_2+1)} s_{(\delta_3-1)}$$

(12)

From the above simple formulas we see that if $S_{\delta}$ is of depth $m$ then it can be expanded as a sum of $2^{m-1}$ terms at the most and each term is a product of a Schur function of depth $m-1$ with a Schur function of depth 1. The above formulas when used in conjunction with the Littlewood’s theorem [9] and Murnaghan formulas [10] simplifies the evaluation of inner product of Schur functions of degree $n$, thus reduces to that Schur functions of degree less than $n$.

Example:

Consider the kronecker product $S_{(n-1,1,1)} \otimes S_{(n-3,2,1)}$.

Using Eq.(12), we get

$$S_{(n-2,1,1)} \otimes S_{(n-3,2,1)} = [S_{(n-2,1)} S_{(1)} - S_{(n-1,1)} S_{(0)}] S_{(n-3,2,1)}$$

(13)

Simplify the equations using Littlewood’s theorem [9] and using the Murnaghan formulas by replacing $n-1$ for $n$, we get

$$S_{(n-2,1,1)} \otimes S_{(n-3,2,1)} = S_{(n-1,1)} + 2 S_{(n-2,2)} + 2 S_{(n-1,1,1)} + 2 S_{(n-3,3)} + 4 S_{(n-3,2,1)} + 2 S_{(n-3,1,1,1)} + 3 S_{(n-4,3,1)} + 2 S_{(n-4,2,2)} + 3 S_{(n-4,2,1,1)} + 2 S_{(n-4,1,1,1,1)} + 3 S_{(n-5,3,2)} + S_{(n-5,3,1,1)} + S_{(n-5,2,2,1)} + S_{(n-5,2,1,1,1)}$$

(14)
4. Conclusion

In this paper, we present a new algorithm for kronecker product of symmetric group representations based on Littlewood’s theorem and Murnaghan’s formulas. We would like to expand our algorithm to find Clebsch Gordan series and Clebsch Gordan coefficients for symmetric groups.

References


Received: November, 2009