On Commutative SCI-Rings and Commutative SCS-Rings

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Abstract. Let $R$ be a commutative ring. An unital $R$-module $M$ is said to have property $(I)$ (resp. property $(S)$) if every injective (resp. surjective) endomorphism of $M$ is an automorphism. The ring $R$ is called commutative $SCI$-ring (resp. $SCS$-ring) if every $R$-module with property$(I)$ (resp. property$(S)$) is finitely cogenerated. In this note we show that the following conditions are equivalent: $(i)$ $R$ is a commutative artinian principal ideal ring; $(ii)$ $R$ is a commutative $SCI$-ring; $(iii)$ $R$ is a commutative $SCS$-ring.

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Let $R$ be a commutative ring with $1 \neq 0$. An $R$-module $M$ is said to have property (I) (resp. property (S)) if every injective (resp. surjective) endomorphism of $R \times M$ is an automorphism of $R \times M$. The characterization of rings for which property (I) (resp. property (S)) characterizes a particular class of the category of $R$-modules has been initiated for the first time in [?] where it has been proved that a commutative ring $R$ is an artinian principal ideal ring if and only if property (I) (resp. property (S)) characterizes the artinian (resp. noetherian) $R$-modules. It is clear that a non necessary commutative ring is a division ring if and only if property (I) (resp. property (S)) characterizes finitely generated free $R$-modules. It has been proved next time in [?] and [?] that a commutative ring is an artinian principal ideal ring if and only if property (I) (resp. property (S)) characterizes finitely generated $R$-modules. Let $R$ be a commutative ring. An $R$-module $M$ is said to be finitely cogenerated if its socle is essential in $M$ and finitely generated; the ring $R$ is called $SCI$-ring (resp. $SCS$-ring) if property (I) (resp. property (S)) characterizes finitely cogenerated modules. The notion of finitely cogenerated modules has been introduced and studied in [?]. In this paper, we show that for a commutative ring $R$ the following conditions are equivalent: (i) $R$ is a artinian principal ideal ring; (ii) $R$ is a $SCI$-ring; (iii) $R$ is a $SCS$-ring. In this note all rings are commutative and associative with $1 \neq 0$ and all modules are unitary. If $R$ is a ring, we note by $J(R)$ or simply by $J$ the Jacobson radical of $R$ and by $rad R$ the prime radical of $R$. The socle of a $R$-module $M$ will be noted $soc(M)$. For all notions not defined in this paper see [?].

1. CONSTRUCTION OF A NON FINITELY COGENERATED
   MODULE WITH BOTH PROPERTY (I) AND PROPERTY (S)
   OVER A LOCAL ARTINIAN RING WHOSE MAXIMAL IDEAL
   IS NOT PRINCIPAL

Let $R$ be a commutative local artinian ring which is a non principal ideal ring. We may suppose without loss of generalities that the ring $R$ is local artinian with Jacobson radical $J = aR + bR$ where $a^2 = b^2 = ab = 0$, $a \neq 0$. 

and \( b \neq 0 \). Following [?] we may write \( R = C \oplus bC \) where \( C \) is an artinian local subring of \( R \) with maximal ideal \( J(C) = aC \neq 0 \). Let \( M \) be the total ring of fractions of the polynomial ring \( C[X] \), \( \sigma \) the endomorphism of the \( C \)-module \( M \) defined by \( \sigma(m) = aXm \) for \( m \in M \); \( \varphi : R \rightarrow \text{End}_CM \) the homomorphism of rings defined by \( \varphi(\alpha + \beta b) = \alpha 1_M + \beta \sigma \) for \( \alpha + \beta b \in R \) where \( \alpha \in C \), \( \beta \in C \) and \( 1_M \) is the identity endomorphism of \( M \). We consider on \( M \) the \( R \)-module structure defined by \( (\alpha + \beta b)m = \varphi(\alpha + \beta b)(m) = \alpha m + \beta aXm \), for \( \alpha + \beta b \in R \), \( (\alpha \text{ and } \beta \in C) \) and for \( m \in M \).

If \( f \) is a \( R \)-endomorphism of the \( R \)-module \( M \), then for all \( m \in M \) we have:

\[
\sigma.f(m) = bf(m) = f(bm) = f(\sigma(m)).
\]

Thus, the \( R \)-endomorphisms of the \( R \)-module \( M \) are the \( C \)-endomorphisms of \( M \) commuting with \( \sigma \).

Following[?] proposition 2.3 the \( R \)-module \( M \) satisfies properties \((I)\) and \((S)\).

For \( d = \lambda a + \gamma b \in J = Ra + Rb \) and for \( am \in aC[X] \) we have:

\[
d.(am) = (\lambda a + \gamma b)am = \varphi(\lambda a + \gamma b)(am) = (\lambda a 1_M + \gamma \sigma)(am) = \lambda a^2 m + \gamma a^2 Xm = 0
\]

Therefore, the submodule \( aC[X] = \bigoplus_{n \geq 0} aCX^n \) of \( M \) is annihilated by \( J \), then \( aC[X] \) is semi-simple as \( R \)-module. Since \( aC[X] \) is not finite length as \( R \)-module, then \( M \) is not finitely cogenerated.

2. CHARACTERIZATION OF COMMUTATIVE SCI-RINGS AND COMMUTATIVE SCS-RINGS

**Proposition 2.1.** Let \( R \) be a commutative SCI-ring(resp. SCS-ring). If \( R \) is an integral domain, then \( R \) is a field.

**Proof.** Let \( K \) be the classical quotient field of the integral domain \( R \). The \( R \)-module \( R_K \) satisfies property \((I)\)(resp. property \((S)\)). Therefore, \( R_K \) is finitely cogenerated. Thus, \( \text{soc}(R_K) \cap R \neq \{0\} \). Let \( S = Ra \ (a \in R \setminus \{0\}) \) a simple submodule of \( \text{soc}(R_K) \cap R \). The map:

\[
\varphi : R \rightarrow S = Ra
\]
$x \mapsto xa$

is an isomorphism of $R$-modules. Therefore $_RR$ is simple. So far $b \in R\setminus\{0\}$ we have $R = Rb = Rb^2$, then $b = cb^2$ for some $c \in R$. It follows that $1 = cb$. □

**Proposition 2.2.** Let $R$ be a commutative SCI-ring (resp. SCS-ring). We have the following results:

1. Every prime ideal of $R$ is a maximal ideal;
2. The Jacobson radical of $R$ is nil;
3. The set of the maximal ideals of $R$ is finite;
4. $R$ is a finite direct product of SCI (resp. SCS) commutative local rings.

**Proof.**

(1) If $p$ is a prime ideal of $R$, then $R/p$ is a commutative integral domain. Since $R/p$ is SCI-ring (resp. SCS-ring), then $R/p$ is a field and $p$ is maximal.

(2) $J = \text{rad}(R)$ is nil results from (1) because $J = J(R) = \text{rad}(R)$.

(3) Let $D$ be the set of all prime ideals of $R$. If $p$ and $p'$ are two elements of $D$ such that $p \not\subseteq p'$ then $\text{Hom}(R/p, R/p') = \{0\}$ (see [?]). Therefore, the semi-simple $R$-module $M = \bigoplus_{p \in D} R/p$ satisfies property (I) (resp. property (S)), so $\text{soc}(M) = M$ is finitely cogenerated. It follows that $M$ is finitely generated and then $D$ is a finite set.

(4) By (3) $R/J \cong \prod_{p \in D} R/p$. So that $R/p$ is a semi-simple ring. Since $J$ is a nil ideal of $R$, then $R$ is semi-perfect. So $R$ is a finite direct product of local SCI-rings (resp. SCS-rings).

□

**Proposition 2.3.** Let $R$ be a commutative SCI-ring (resp. SCS). Then $R$ is Artinian.

**Proof.** Following [?](10.8) it is enough to show that every finitely generated $R$-module is finitely cogenerated. Since $R$ is a commutative ring and every prime ideal of $R$ is maximal, then every finitely generated $R$-module $M$ satisfies property (I) (resp. property (S)) (see [?] resp[?]), and from the hypothesis that $R$ is a SCI-ring (resp. SCS-ring), it follows that $M$ is finitely cogenerated. □

**Proposition 2.4.** Let $R$ be a commutative SCI (resp. SCS) ring. Then $R$ is a finite direct product of local artinian SCI (resp. SCS)-ring.
Proof. It follows from (2.2) and (2.3).

**Theorem 2.5.** Let R be a commutative ring. Then the following conditions are equivalent:

1. R is a SCI-ring;
2. R is a SCS-ring;
3. R is an artinian principal ideal ring.

Proof. (1) ⇒ (3)(resp.(2)) ⇒ (3)
Following (2.4) we may suppose that R is a commutative local artinian SCI (resp. SCS-ring). Then by §1 R is an principal ideal ring.

(3) ⇒ (1)(resp.(3) ⇒ (2))
Following [?], every R-module satisfying property (I) (resp. property (S)) is finitely generated, then finitely cogenerated (see.[?]((10.18))). Therefore R is a SCI-ring(resp. SCS-ring).

**References**


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