McCoy Rings Relative to a Monoid

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Abstract

For a monoid $M$, we introduce $M$-McCoy rings, which are a generalization of McCoy rings, and investigate their properties. Every reversible ring is $M$-McCoy for any unique product monoid $M$. It is also shown that a finitely generated Abelian group $M$ is torsion free if and only if there exists a ring $R$ such that $R$ is $M$-McCoy.

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1 Introduction

Throughout in this paper, all rings are associative with identity. A ring $R$ is right McCoy ring if the equation $f(x)g(x) = 0$ where $f(x), g(x) \in R[x]\{0\}$, implies that there exists $s \in R\{0\}$ such that $f(x)s = 0$. Left McCoy rings are defined similarly[5]. McCoy rings were chosen because McCoy [4] had noted that every commutative rings satisfy this condition.

Let $M$ be a monoid. In the following, $e$ will always stand for the identity of $M$. A ring $R$ is called a right $M$-McCoy ring(a McCoy ring relative to $M$), if whenever elements $\alpha = a_1g_1 + \ldots + a_ng_n, \beta = b_1h_1 + \ldots + b_mh_m \in R[M]$ satisfy $\alpha\beta = 0$, then $\alpha s = 0$ for some $s \in R\{0\}$. Left $M$-McCoy rings are defined similarly. If $M = \{e\}$ then every ring is $M$-McCoy. Let $M = (N\cup\{0\}, +)$. Then a ring $R$ is $M$-McCoy if and only if $R$ is McCoy ring. The following results will give more examples of $M$-McCoy rings.
Recall that a monoid \( M \) is called an \( u.p.-\) monoid (unique product monoid) if for any two nonempty finite subsets \( A, B \subseteq M \) there exist an element \( g \in M \), uniquely presented in the form \( ab \) where \( a \in A \) and \( b \in B \). The class of \( u.p.-\)monoids is quite large and important (see [1]). For example, this class includes the right or left ordered monoids, submonoids of a free groups, and torsion free nilpotent group. Every \( u.p.-\)monoid \( M \) has no nonunity of finite order.

For notation \( M_n(R), T_n(R), I_n \) and \( e_{ij}, 1 \leq i, j \leq n \) denote the \( n \times n \) matrix ring over \( R \), the upper triangular matrix ring over \( R \), the identity matrix and the matrix with 1 at \((i, j)-\)entry and 0 elsewhere, respectively.

2 Main results

**Definition 2.1** Let \( M \) be a monoid. A ring \( R \) is called right \( M\)-McCoy ring if whenever elements \( \alpha = a_1g_1 + ... + a_ng_n, \beta = b_1h_1 + ... + b_mh_m \in R[M] \) satisfy \( \alpha\beta = 0 \), then \( \alpha s = 0 \) for some \( s \in R \setminus \{0\} \). Left \( M\)-McCoy rings are defined analogously.

Recall that a ring \( R \) is called reversible if \( ab = 0 \) then \( ba = 0 \) for all \( a, b \in R \).

**Proposition 2.2** Let \( M \) be an \( u.p.-\)monoid and \( R \) a reversible ring. Then \( R \) is \( M\)-McCoy.

**proof:** Let \( \alpha = a_1g_1 + ... + a_ng_n \) and \( \beta = b_1h_1 + ... + b_mh_m \in R[M] \) be such that \( \alpha \beta = 0 \). We claim that there exists \( s \in R \setminus \{0\} \) such that \( \alpha s = 0 \). We proceed by induction on \( n \).

Let \( n = 1 \). Then \( \alpha = a_1g_1 \). By [1, Lemma 1.1] \( g_1h_i \neq g_1h_j \) for \( i \neq j \). Thus \( a_1b_j = 0 \) for all \( j \).

Let \( n \geq 2 \). Since \( M \) is \( u.p.-\)monoid, there exist \( i, j \) with \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \) such that \( g_ih_j \) is uniquely presented by considering two subset \( A = \{g_1, g_2, ..., g_n\} \) and \( B = \{h_1, h_2, ..., h_m\} \) of \( M \). We may assume, without loss of generality, that \( i = 1, j = 1 \). Thus \( a_1b_1g_1h_1 = 0 \) and hence \( a_1b_1 = 0 \). Since \( R \) is reversible, \( b_1a_1 = 0 \) follows. Thus \( b_1\alpha \beta = 0 \) and so

\[
\left( \sum_{i=2}^{n} (b_1a_i)g_i \right) \left( \sum_{j=1}^{m} b_jh_j \right) = 0.
\]

By induction, we have \( \left( \sum_{i=2}^{n} (b_1a_i)g_i \right) s = 0 \) for some \( s \in R \setminus \{0\} \). Thus \( \alpha sb_1 = 0 \). This prove the results. Similarly, by induction on \( m \) we prove that \( R \) is left \( M\)-McCoy.
Let \((M, \leq)\) be an ordered monoid. If for any \(g, \dot{g}, h \in M,\) \(g < \dot{g}\) implies \(gh < \dot{gh}\) and \(hg < h\dot{g}\), then \((M, \leq)\) is called a strictly ordered monoid.

**Corollary 2.3** Let \(M\) be a strictly totally ordered monoid and \(R\) a reversible ring. Then \(R\) is \(M\)-McCoy.

**Corollary 2.4** Let \(R\) be a reversible ring. Then \(R\) is \(Z\)-McCoy, that is, for any \(\alpha = \sum a_{ng-n} + \ldots + a_qg_q, \beta = \sum b_{mh-m} + \ldots + b_ph_p \in R[x, x^{-1}],\) if \(\alpha\beta = 0,\) then there exist \(t, s \in R \setminus \{0\}\) such that \(\alpha s = 0\) and \(t\beta = 0.\)

Taking \(M = (N \cup \{0\}, +)\) in Corollary 2.3, it follows that every reversible ring is McCoy; [5, Theorem 2].

**Proposition 2.5** Let \(M\) be a monoid with \(|M| \geq 2\). Then a ring \(R\) is the right (resp., left) \(M\)-McCoy if and only if the ring

\[
R_n := \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}
\]

is right (resp., left) \(M\)-McCoy for any \(n \geq 1.\)

**proof:** Let \(R\) be an \(M\)-McCoy ring and \(\alpha = A_1g_1 + \ldots + A_ng_n, \beta = B_1h_1 + \ldots + B_mh_m \in R_n[M].\) Assume that \(\alpha\beta = 0.\) Let

\[
A_i = \begin{pmatrix} a^i & a_{i1}^i & \cdots & a_{1n}^i \\ 0 & a^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^i \end{pmatrix},
B_j = \begin{pmatrix} b^j & b_{j1}^j & \cdots & b_{1n}^j \\ 0 & b^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b^j \end{pmatrix}.
\]

It is easy to see that there exists an isomorphism \(R_n[M] \cong (R[M])_n\) defined by

\[
\sum_{k=1}^s \begin{pmatrix} a^k & a_{12}^k & \cdots & a_{1n}^k \\ 0 & a^k & \cdots & a_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^k \end{pmatrix} g_k \rightarrow \begin{pmatrix} \sum_{k=1}^s a^k g_k \\ \sum_{k=1}^s a_{12}^k g_k \\ \vdots \\ \sum_{k=1}^s a_{2n}^k g_k \end{pmatrix},
\]

\((*)\).
Thus

\[
\alpha = \begin{pmatrix}
\alpha_1 & \alpha_{12} & \cdots & \alpha_{1n} \\
0 & \alpha_1 & \cdots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_1
\end{pmatrix}, \quad \beta = \begin{pmatrix}
\beta_1 & \beta_{12} & \cdots & \beta_{1n} \\
0 & \beta_1 & \cdots & \beta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta_1
\end{pmatrix}
\]

are elements of \((R[M])_n\) such that \(\alpha \beta = 0\).

Thus \(\alpha_1 \beta_1 = 0\). Since \(R\) is right \(M\)-McCoy, there exists \(s \in R \setminus \{0\}\) such that \(\alpha_1 s = 0\). Therefore, \(\alpha (se_{1n}) = 0\) and it follows that \(R_n\) is \(M\)-McCoy.

Conversely, let \(R_n\) be \(M\)-McCoy ring and \(\alpha = a_1g_1 + \ldots + a_ng_n\), \(\beta = b_1h_1 + \ldots + b_mh_m\) satisfy \(\alpha \beta = 0\) in \(R[M]\). From the isomorphism (*) we have,

\[
\begin{pmatrix}
\alpha & 0 & \cdots & 0 \\
0 & \alpha & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha
\end{pmatrix}
\begin{pmatrix}
\beta & 0 & \cdots & 0 \\
0 & \beta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \beta
\end{pmatrix} = 0.
\]

Then as in proof of another side there exists \(s \in R \setminus \{0\}\) such that \(as = 0\). This proves the results.

A ring \(R\) is called \(M\)-Armendariz ring, if whenever elements \(\alpha = a_1g_1 + \ldots + a_ng_n\), \(\beta = b_1h_1 + \ldots + b_mh_m \in R[M]\) satisfy \(\alpha \beta = 0\), then \(a_ib_j = 0\) for each \(i, j\). \(M\)-Armendariz rings are \(M\)-McCoy by definition. However there exists an \(M\)-McCoy ring which is not \(M\)-Armendariz. If \(R\) is an \(M\)-McCoy ring, then \(R_4\) is \(M\)-McCoy too, by Proposition 2.5. However \(R_4\) is not \(M\)-Armendariz by [7, Remark 1.8].

Based on Proposition 2.5, we may suspect \(M_n(R)\) or \(T_n(R)\) over an \(M\)-McCoy ring is still \(M\)-McCoy. But the following example erases the possibility.

**Example 2.6** Let \(M\) be a monoid with \(|M| \geq 2\) and \(R\) a ring. Take \(e \neq g \in M\). Let \(A = C = e_{12}, B = e_{11}, D = -e_{22}\),

\[
\alpha_1 = \begin{pmatrix}
A & 0 \\
0 & 0
\end{pmatrix} e + \begin{pmatrix}
B & 0 \\
0 & 0
\end{pmatrix} g
\]

\[
\beta_1 = \begin{pmatrix}
C & 0 \\
0 & 0
\end{pmatrix} e + \begin{pmatrix}
A & 0 \\
0 & I_{n-2}
\end{pmatrix} g
\]

\[
\beta_2 = \begin{pmatrix}
C & 0 \\
0 & 0
\end{pmatrix} e + \begin{pmatrix}
D & 0 \\
0 & 0
\end{pmatrix} g
\]
McCoy rings relative to a monoid

\[
\alpha_2 = \begin{pmatrix} A & 0 \\ 0 & I_{n-2} \end{pmatrix} e + \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} g
\]

are elements of \(M_n(R)[M]\) so \(\alpha_1\beta_1 = 0\) and \(\alpha_2\beta_2 = 0\). But if \(S\beta_1 = 0\) or \(\alpha_2T = 0\) for some \(S,T \in M_n(R)[M]\setminus\{0\}\), then \(S = T = 0\), thus \(M_n(R)\) is neither left nor right M-McCoy.

It is natural to ask whether \(R\) is an M-McCoy ring for a monoid \(M\), if for any nonzero proper ideal \(I\) of \(R\), \(R/I\) are M-McCoy, where \(I\) is considered as a M-McCoy ring without identity. However, we have a negative answer to this question by the following example.

Example 2.7 Let \(F\) be a field and consider \(R = T_2(F)\), which is not M-McCoy, for u.p.- monoid \(M\) by Example 2.6. Next we show that \(R/I\) and \(I\) are M-McCoy for some nonzero proper ideal \(I\) of \(R\). Note that the only nonzero proper ideal of \(R\) are \(\begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}\), \(\begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix}\).

Let \(I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}\). Then \(R/I \cong F\) and so \(R/I\) is M-McCoy.

Now let \(\alpha = \sum_{i=1}^{n} \left(\begin{array}{cc} a_i & b_i \\ 0 & 0 \end{array}\right) g_i\) and \(\beta = \sum_{j=1}^{m} \left(\begin{array}{cc} c_j & d_j \\ 0 & 0 \end{array}\right) h_j\) be nonzero elements of \(I[M]\) such that \(\alpha\beta = 0\). From the isomorphism \(T_2(F)[M] \cong T_2(F[M])\) defined by:

\[
\sum_{i=1}^{n} \left(\begin{array}{cc} a_i & b_i \\ 0 & c_i \end{array}\right) g_i \rightarrow \left(\begin{array}{c} \sum_{i=1}^{n} a_i g_i \\ \sum_{i=1}^{n} b_i g_i \\ \sum_{i=1}^{n} c_i g_i \end{array}\right)
\]

we have \(\alpha_1\beta_1 = \alpha_1\beta_2 = 0\) (*), where \(\alpha_1 = \sum_{i=1}^{n} a_i g_i\), \(\beta_1 = \sum_{j=1}^{m} c_j h_j\) and \(\beta_2 = \sum_{j=1}^{m} c_j h_j \in F[M]\). If \(\alpha_1 = 0\), then \(\alpha e_{11} = 0\). Suppose \(\alpha_1 \neq 0\). Since \(\beta \neq 0\), then \(\beta_1 \neq 0\) or \(\beta_2 \neq 0\). From the equation (*) and the condition that \(F\) is right M-McCoy, we have \(\alpha_1 s = 0\) for some nonzero \(s \in F\), therefore \(\alpha(se_{11}) = 0\). Thus \(I\) is right M-McCoy, and \(I\) is left M-McCoy since \(e_{12}\beta = 0\).

Proposition 2.8 Let \(M\) be a cancelative monoid and \(N\) is an ideal of \(M\). If \(R\) is right \(N\)-McCoy, then \(R\) is right \(M\)-McCoy.

**Proof**: Suppose that \(\alpha = a_1 g_1 + ... + a_n g_n\), \(\beta = b_1 h_1 + ... + b_m h_m \in R[M]\) satisfy \(\alpha\beta = 0\). Take \(g \in N\), then \(g g_1, ..., g g_n, h_1 g, ..., h_m g \in N\) and \(g g_i \neq g g_j\).
and \( h_i g \neq h_j g \) when \( i \neq j \). Now from
\[
\left( \sum_{i=1}^{n} a_i gg_i \right) \left( \sum_{j=1}^{m} b_j h_j g \right) = 0
\]
and from the hypothesis, there exists \( s \in R \setminus \{0\} \) such that \( (\sum_{i=1}^{n} a_i gg_i) s = 0 \). Thus \( \alpha s = 0 \), it follows that \( R \) is \( M \)-McCoy.

**Lemma 2.9** Let \( M \) be a cyclic group of order \( n \geq 2 \) and \( R \) a ring with \( 0 \neq 1 \), then \( R \) is not \( M \)-McCoy.

**Proof:** Suppose \( M = \{e, g, g^2, \ldots, g^{n-1}\} \). Let \( \alpha = 1e + 1g + 1g^2 + \ldots + 1g^{n-1} \) and \( \beta = 1e + (-1)g \). Then \( \alpha \beta = 0 \), but if \( \alpha s = 0 \) then \( s = 0 \). Thus \( R \) is not \( M \)-McCoy.

**Lemma 2.10** Let \( M \) be a monoid and \( N \) a submonoid of \( M \). If \( R \) is \( M \)-McCoy then \( R \) is \( N \)-McCoy.

Let \( T(G) \) be the set of elements of finite order in an abelian group \( G \). Then \( G \) is said to be torsion free if \( T(G) = \{e\} \).

**Theorem 2.11** Let \( G \) be a finitely generated abelian group. Then the following conditions on \( G \) are equivalent.

(1) \( G \) is torsion free.
(2) There exists a ring \( R \) with \( |R| \geq 2 \) such that \( R \) is \( G \)-McCoy.

**Proof:** (2) \( \Rightarrow \) (1) If \( g \in T(G) \) and \( g \neq e \), then \( N = \langle g \rangle \) is cyclic group of finite order. If a ring \( R \neq \{0\} \) is \( G \)-McCoy, then by Lemma 2.10 \( R \) is \( N \)-McCoy, a contradiction with Lemma 2.9. Thus every ring \( R \neq \{0\} \) is not \( G \)-McCoy.

(1) \( \Rightarrow \) (2). If \( G \) is finitely generated abelian group with \( T(G) = \{e\} \), then \( G \cong Z \times Z \times \ldots \times Z \), a finite direct product group \( Z \), by [7, Lemma 1.13], \( G \) is u.p.-monoid. Let \( R \) be a commutative ring then by Proposition 2.2, \( R \) is \( G \)-McCoy.

A classical right quotient ring for a ring \( R \) is a ring \( Q \) which contains \( R \) as a subring in such a way that every regular element (i.e., non-zero-divisor) of \( R \) is invertible in \( Q \) and \( Q = \{ab^{-1} : a, b \in R, b \text{ is regular}\} \). A ring \( R \) is called right Ore if given \( a, b \in R \) with \( b \) regular there exist \( a_1, b_1 \in R \) with \( b_1 \) regular
such that \(ab_1 = ba_1\). Classical left quotient rings and left Ore rings are defined similarly. It is a well-known fact that \(R\) is a right (resp., left) Ore ring if and only if the classical right (resp., left) quotient ring of \(R\) exists.

**Theorem 2.12** Suppose that there exists the classical right quotient ring \(Q\) of a ring \(R\). Then \(R\) is right M-McCoy if and only if \(Q\) is right M-McCoy.

**proof:** " \(\Rightarrow\) " Let \(A = \sum_{i=1}^{n} \alpha_i g_i\) and \(B = \sum_{j=1}^{m} \beta_j h_j\) be nonzero elements of \(R[M]\) such that \(AB = 0\). Since \(Q\) is the classical right quotient ring, we may assume that \(\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}\) with \(a_i, b_j \in R\) for all \(i, j\) and regular elements \(u, v \in R\). For each \(j\), there exists \(c_j \in R\) and a regular element \(w \in R\) such that \(u^{-1} b_j = c_j w^{-1}\). Denote \(A_1 = \sum_{i=1}^{n} a_i g_i\) and \(B_1 = \sum_{j=1}^{m} c_j h_j\). Then the equation \(A_1 B_1 (vw)^{-1} = 0\), thus there exist a nonzero element \(s \in R\{0\}\) such that \(As = 0\), then \(\alpha_i (us) = 0\) for every \(i\). It implies that \(A(us) = 0\) and \(us\) is nonzero element of \(Q\). Hence \(Q\) is right M-McCoy.

" \(\Leftarrow\) " Let \(\alpha = \sum_{i=1}^{n} \alpha_i g_i, \beta = \sum_{j=1}^{m} b_j h_j \in R[M]\) such that \(\alpha \beta = 0\). Then there exists a nonzero element \(K \in Q\) such that \(\alpha K = 0\) since \(Q\) is right M-McCoy. Because \(Q\) is a classical right quotient ring, we can assume \(K = au^{-1}\) for some \(a \in R\{0\}\) and regular element \(u\). Then \(\alpha au^{-1} = \alpha K = 0\) implies that \(\alpha a = 0\). Therefore, \(R\) is a right M-McCoy ring.

By the Goldie Theorem, if \(R\) is semiprime left and right Goldie ring, then \(R\) has the classical left and right quotient ring. Hence there exists a class of rings satisfying the following hypothesis.

**Corollary 2.13** Suppose that there exists the classical left and right quotient ring \(Q\) of a ring \(R\). Then \(R\) is M-McCoy if and only if \(Q\) is M-McCoy.

In following proposition we consider the case of direct limit of direct system of M-McCoy rings.

**Proposition 2.14** The direct limit of a direct system of M-McCoy rings is also M-McCoy.

**proof:** Let \(D = \{R_i, \alpha_{ij}\}\) be a direct system of M-McCoy rings \(R_i\) for \(i \in I\) and ring homomorphisms \(\alpha_{ij} : R_i \to R_j\) for each \(i \leq j\) satisfying \(\alpha_{ij}(1) = 1\), where \(I\) is directed partially ordered set. Set \(R = \text{lim} R_i\) be direct
limit of $D$ with $l_i : R_i \to R$ and $l_j \alpha_{ij} = l_i$. We will prove that $R$ is M-McCoy ring. Take $x, y \in R$, then $x = l_i(x_i), y = l_j(y_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define $x + y = l_k(\alpha_{ik}(x_i) + \alpha_{jk}(y_j))$ and $xy = l_k(\alpha_{ik}(x_i)\alpha_{jk}(y_j))$, where $\alpha_{ik}(x_i)$ and $\alpha_{jk}(y_j)$ are in $R_k$. Then $R$ forms a ring with $l_i(0) = 0$ and $l_i(1) = 1$.

Now suppose $AB = 0$ for $A = \sum_{s=1}^{m}a_s g_s, B = \sum_{t=1}^{n}b_t h_t$ in $R[M]\{0\}$. There exist $i_s, j_t, k \in I$ such that $a_s = l_{i_s}(a_{i_s}), b_t = l_{j_t}(b_{j_t}), i_s \leq k, j_t \leq k$. So $a_s b_t = l_k(\alpha_{i_s,k}(a_{i_s})\alpha_{j_t,k}(b_{j_t}))$.

Thus $AB = (\sum_{s=1}^{m}l_k(\alpha_{i_s,k}(a_{i_s}))g_s)(\sum_{t=1}^{n}l_k(\alpha_{j_t,k}(b_{j_t}))h_t) = 0$. But $R_k$ is M-McCoy ring and so there exist $0 \neq d \in R_k$ such that $Al_k(d) = 0$. Thus $R$ is M-McCoy ring.

References


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