Characterizability of Finite Simple Groups
by their Order Components:
A Summary of Results

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Abstract

Let $G$ be a finite group and let $OC(G)$ be the set of order components of $G$. The number of non-isomorphic classes of finite groups $H$ satisfying $OC(G) = OC(H)$ is denoted by $h(G)$. If $h(G) = k$ then $G$ is called a $k$-recognizable group by the set of its order components and if $k = 1$, $G$ is called a recognizable group. The main consequence of recognizability of a group $G$ by its order components is the validity of Thompson’s conjecture for $G$. In this paper we consider recognizability of simple groups with exactly two components and we survey currently known results in this regard which shows that the recognition problem for these groups is completely solved.

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1 Introduction

Let $n$ be a positive integer. The set of all prime divisors of $n$ is denoted by $\pi(n)$ and if $G$ is a finite group, then we set $\pi(G) = \pi(|G|)$. The Gruenberg-Kegel graph of $G$, or the prime graph of $G$, is denoted by $GK(G)$ and is defined as follows.
The vertex set of $GK(G)$ is $\pi(G)$ and two distinct vertices $p$ and $q$ are adjacent
if and only if $G$ contains an element of order $pq$. We assume that $\{\pi_i \mid 1 \leq i \leq s(G)\}$ is the set of connected components of $GK(G)$. Also, we adopt the assumption that if the order of $G$ is even, then the notation is chosen so that $2 \in \pi_1$. Clearly, the order of $G$ can be expressed as the product of the numbers $m_1, m_2, \ldots, m_{s(G)}$ where $\pi(m_i) = \pi_i$, $1 \leq i \leq s(G)$. Based on these notations, if the order of $G$ is even and $s(G) \geq 2$, then $m_2, \ldots, m_{s(G)}$ are odd numbers. The positive integers $m_1, m_2, \ldots, m_{s(G)}$ are called the order components of $G$ and the set $OC(G) = \{m_1, m_2, \ldots, m_{s(G)}\}$ is called the set of order components of $G$.

**Definition 1.** Let $G$ be a finite group. The number of non-isomorphic finite groups with the same order components as $G$ is denoted by $h(G)$. If $h(G) = k$, then $G$ is called a $k$-recognizable group by the set of its order components and if $k = 1$, $G$ is simply called a recognizable group.

Obviously, for any finite group $G$ we have $h(G) \geq 1$. The components of the Gruenberg-Kegel graph of $GK(P)$ of any non-Abelian finite simple group $P$ with $GK(P)$ disconnected are found in [27] and [30], from which we can find the order components of $P$.

A motivation for characterizing finite groups by the set of their order components is the following conjecture of J. G. Thompson.

**Conjecture (Thompson [2])** For a finite group $G$ let

$$N(G) = \{n \in \mathbb{N} \mid G \text{ has a conjugacy class of size } n\}.$$  

If $Z(G) = 1$ and $M$ is a non-Abelian finite simple group satisfying $N(G) = N(M)$, then $G \cong M$.

## 2 Preliminary results

The structure of a finite group with disconnected Gruenberg-Kegel graph is determined by [30] as follows.

**Proposition 1.** [30] Let $G$ be a finite group with $s(G) \geq 2$. Then one of the following holds:

- (1) $G$ is either a Frobenius or a $2$–Frobenius group.
• (2) $G$ has a normal series $1 \leq H \leq K \leq G$ such that $H$ is a nilpotent $\pi_1$-group, $\frac{K}{H}$ is a non-Abelian simple group, $\frac{G}{K}$ is a $\pi_1$-group, $|\frac{G}{K}|$ divides $\text{Out}(\frac{K}{H})$ and any odd order component of $G$ is equal to one of the odd order components of $K$.

Proposition 2. \[30\]

• (a) Let $G$ be a Frobenius group of even order with kernel and components $K$ and $H$, respectively. Then $s(G) = 2$ and the components of $GK(G)$ are $\pi(H)$ and $\pi(K)$.

• (b) Let $G$ be a 2-Frobenius group of even order, then $s(G) = 2$ and $G$ has a normal series $1 \leq H \leq K \leq G$ such that $|\frac{K}{H}| = m_2$, $|H| = m_1$ and $|\frac{G}{K}|$ divides $|\frac{K}{H}| - 1$ and $H$ is a nilpotent $\pi_1$-group.

Proposition 3. \[30\] Let $G$ be a finite group with $s(G) \geq 2$. If $H \leq G$ is a $\pi_1$-group, then

$$\prod_{j=1, j \neq i}^{s(G)} (|H| - 1).$$

By Proposition 3, if $H$ is a $\pi_1$-subgroup of $G$, $H \leq G$, and $s(G) = 2$, then $m_2(|H| - 1)$, implying $|H| \equiv 1(\text{mod } m_2)$. The following result of Zsigmondy is used to prove the main theorem of [31].

Proposition 4. Let $n$ and $a$ be integers greater than 1. Then there exists a prime divisor $p$ does not divide $a^i - 1$ for all $i, 1 \leq i \leq n$, except in the following cases:

• (1) $n = 2, a = 2^k - 1$, where $k \geq 2$.

• (2) $n = 6, a = 2$.

The prime $p$ in Proposition 4 is called a Zsigmondy prime for $a^n - 1$. In [2], it is proved that if $s(G) \geq 3$, then Thompson’s conjecture holds. It is also proved that if $G$ and $M$ are finite groups with $s(M) \geq 2$, $Z(G) = 1$ and $N(G) = N(M)$, then $|G| = |M|$, in particular $s(M) = s(G)$ and $OC(G) = OC(M)$. Therefore if the simple group $M$ with disconnected prime group is characterizable by the set of its order components, then Thompson’s conjecture holds for $M$.

Therefore, in order to complete the proof of Thompson’s conjecture for the simple groups $P$ with $s(P) \geq 2$, it is enough to prove Thompson’s conjecture for $s(P) = 2$. 
In [24], it is proved that if \( n = 2^m \geq 4 \) and \( q \) is an odd prime power, then 
\[
\begin{align*}
h(C_n(q)) = h(B_n(q)) &= 2 \\
h(C_n(q)) &= 1.
\end{align*}
\] 
In [11] it is shown that \( h(B_p(3)) = h(C_p(3)) = 2 \) where \( p \) is an odd prime number, and apart these exceptional cases the rest of simple groups \( P \) with \( s(P) = 2 \) are recognizable by the set of their order components.

In a series of articles [4, 5, 6, 29] it is proved that the sporadic groups, and finite groups \( PSL_2(q), \ 3D_4(q), \ 2D_n(3); \ (9 \leq n = 2^m + 1 \text{ not a prime}) \) and \( 2D_{p+1}(q); \ (5 < p \neq 2^m - 1) \) are characterized by the order components of their prime graph.

The recognizability of groups \( L_{p+1}(2), \ 2D_p(3) \) (where \( p \geq 5 \) is a prime number not of the form \( 2^m + 1 \)), \( 2D_n(2) \) (where \( n = 2^m + 1 \geq 5 \)), \( D_{p+1}(2), \ D_{p+1}(3) \) and \( D_p(q) \) (where \( p \geq 5 \) is a prime number and \( q = 2, 3 \) or 5) are proved by M. R. Darafsheh et. al. in [7, 8, 9, 10, 12].

Also, characterizability of \( E_6(q), \ 2E_6(q), \ 2D_n(q) \) (where \( n = 2^m \)), \( PSL(p,q), PSU(p,q), PSL(p+1,q), PSU(p+1,q), PSL(3,q) \) where \( q \) is an odd prime power, \( PSL(3,q) \) for \( q = 2^n \) and \( PSU(3,q) \) for \( q > 5 \) by their order components are proved in a series of articles by B. Khosravi et. al. [18, 19, 20, 22, 23, 25, 26, 14, 15, 16]. In addition, \( r-\)recognizability of \( B_n(q) \) and \( C_n(q) \) (where \( n = 2^m \geq 4 \)) are proved in [24].

The following open problem contains all remaining cases to prove that all simple non-Abelian groups, as \( P \), with \( s(P) = 2 \) are characterizable by order components.

**Open Problem.** Are the groups \( F_4(q) \) (\( q \) odd), \( G_2(q) \) (\( 2 < q \equiv \pm 1(\text{mod } 3) \)) and \( C_p(2) \) characterizable by their order components?

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**Table.** The simple non-Abelian groups $P$ with $s(P) = 2$ and results about their characterizability by order components ($p$ is a prime number)

<table>
<thead>
<tr>
<th>Group</th>
<th>Description</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$, $6 &lt; n = p, p + 1, p + 2$ one of $n, n - 2$ is not a prime</td>
<td>[1]</td>
<td></td>
</tr>
<tr>
<td>$A_{p-1}(q)(p, q) ≠ (3, 2), (3, 4)$</td>
<td>[22]</td>
<td></td>
</tr>
<tr>
<td>$A_p(q)(q - 1)</td>
<td>(p + 1)$</td>
<td>[25],[7]</td>
</tr>
<tr>
<td>$2A_{p-1}(q)$</td>
<td></td>
<td>[23]</td>
</tr>
<tr>
<td>$2A_{p}(q+1)</td>
<td>(p + 1), (p, q) ≠ (3, 3), (5, 2)$</td>
<td>[26]</td>
</tr>
<tr>
<td>$B_n(q), n = 2^m \text{ q odd}, B_p(3)$, $C_n(q), n = 2^m, C_p(3)$</td>
<td>[11],[24]</td>
<td></td>
</tr>
<tr>
<td>$C_p(2)$</td>
<td>unknown</td>
<td></td>
</tr>
<tr>
<td>$D_{p+1}(q), q = 2, 3$</td>
<td>[10]</td>
<td></td>
</tr>
<tr>
<td>$2D_n(q), n = 2^m$</td>
<td></td>
<td>[20]</td>
</tr>
<tr>
<td>$2D_n(2), n = 2^m + 1 ≥ 5$</td>
<td></td>
<td>[9]</td>
</tr>
<tr>
<td>$2D_p(3), p ≠ 2^m + 1 ≥ 5$</td>
<td></td>
<td>[8]</td>
</tr>
<tr>
<td>$2D_n(3), 9 ≤ n = 2^m + 1 ≠ \text{prime}$</td>
<td></td>
<td>[6]</td>
</tr>
<tr>
<td>$G_2(q), 2 &lt; q ≡ ±1(mod 3)$</td>
<td>unknown</td>
<td></td>
</tr>
<tr>
<td>$3D_4(q)$</td>
<td></td>
<td>[5]</td>
</tr>
<tr>
<td>$F_4(q), q \text{ odd}$</td>
<td>unknown</td>
<td></td>
</tr>
<tr>
<td>$E_6(q)$</td>
<td></td>
<td>[18]</td>
</tr>
<tr>
<td>$2E_6(q)$</td>
<td></td>
<td>[19]</td>
</tr>
<tr>
<td>$2D_{p+1}(2), 5 ≤ p ≠ 2^m - 1$</td>
<td></td>
<td>[29]</td>
</tr>
<tr>
<td>$D_p(q), g = 2, 3, 5$</td>
<td></td>
<td>[12]</td>
</tr>
<tr>
<td>$M_{12}, J_2, Ru, He, Mcl, Co_1, Co_3, Fi_{22}, HN$</td>
<td></td>
<td>[3]</td>
</tr>
</tbody>
</table>
References


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