On Hyper $BCK$-Algebras with Condition R.M.

Habib Harizavi

harizavi@scu.ac.ir

Abstract

In this paper, we introduce and prove some properties of the hyper $BCK$-algebras satisfying the implication

$$(\forall x, y, z \in H)(x \trianglelefteq y \implies x \circ z \trianglelefteq y \circ z).$$

Such hyper $BCK$-algebra is called the hyper $BCK$-algebra with condition R.M. We investigate the generated (weak)hyper $BCK$-ideals of the hyper $BCK$-algebras with condition R.M. and show that every weak hyper $BCK$-ideal is a hyper $BCK$-ideal. Also, we give a characterization of the elements of the generated hyper $BCK$-ideals.

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Keywords: hyper $BCK$-algebra, (weak)hyper $BCK$-ideal, generated hyper $BCK$-ideal, condition R.M.

1 Introduction

The study of $BCK$-algebras was initiated by Y. Imai and K. Iséki[4] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of $BCK$-algebras. The hyper structure theory (called also multi algebras) was introduced in 1934 by F. Marty at the 8th congress of Scandinavian Mathematicians. In [5], Y.B. Jun et al. applied the hyper structures to $BCK$-algebras, and introduced the notion of a hyper $BCK$-algebra which is a generalization of $BCK$-algebra, and investigated some related properties. Now, we follow [1] and [5] and introduce the hyper $BCK$-algebra which satisfies the implication $\forall x, y, z \in H(x \trianglelefteq y \implies x \circ z \trianglelefteq y \circ z)$ and investigate some properties of it. In such hyper $BCK$-algebra, we study the generated hyper $BCK$-ideals and characterize its elements.
2 Preliminary Notes

Let \( H \) be a non-empty set endowed with a hyper operation \( \circ \) , that is, \( \circ \) is a function from \( H \times H \) to \( P^*(H) = P(H) \backslash \{\phi\} \). For two subset \( A \) and \( B \) of \( H \), denote by \( A \circ B \) the set \( \bigcup_{a \in A, b \in B} a \circ b \). We shall use \( x \circ y \) instead of \( x \circ \{y\}, \{x\} \circ y \), or \( \{x\} \circ \{y\} \).

**Definition 2.1** [5] By a hyper \( BCK \)-algebra we mean a non-empty set \( H \) endowed with a hyper operation \( \circ \) and a constant \( 0 \) satisfying the following axioms: for all \( x, y, z \in H \),

- \((H1)\) \((x \circ z) \circ (y \circ z) \leq x \circ y\),
- \((H2)\) \((x \circ y) \circ z = (x \circ z) \circ y\),
- \((H3)\) \(x \circ H \leq \{x\}\),
- \((H4)\) \(x \leq y\) and \( y \leq x\) imply \( x = y\),

where \( x \leq y\) is defined by \( 0 \in x \circ y\) and for every \( A, B \subseteq H \), \( A \leq B\) is defined by \( \forall a \in A, \exists b \in B\) such that \( a \leq b\). In such case, we call \( \leq \) the hyper order in \( H\).

**Theorem 2.2** [5, 5] In any hyper \( BCK\)-algebra \( H \), the following hold: for any \( x, y, z \in H \) and \( A, B \subseteq H \),

- \((a1)\) \(0 \circ 0 = \{0\}\),
- \((a2)\) \(0 \leq x\),
- \((a3)\) \(x \leq x\),
- \((a4)\) \(A \leq A\),
- \((a5)\) \(A \leq 0\) implies \( A = \{0\}\),
- \((a6)\) \(A \subseteq B\) implies \( A \leq B\),
- \((a7)\) \(0 \circ x = \{0\}\),
- \((a8)\) \(x \circ y \leq x\),
- \((a9)\) \(x \circ 0 = \{x\}\),
- \((a10)\) \(y \leq z\) implies \( x \circ z \leq x \circ y\),
- \((a11)\) \(x \circ y = \{0\}\) implies \( (x \circ z) \circ (y \circ z) = \{0\}\) and \( x \circ z \leq y \circ z\),
- \((a12)\) \(A \circ \{0\} = \{0\}\) implies \( A = \{0\}\).

**Definition 2.3** [5] Let \( H \) be a hyper \( BCK\)-algebra. Then a non-empty subset \( S \) of \( H \) is called a hyper subalgebra of \( H \) if \( S \) is a hyper \( BCK\)-algebra with respect to the hyper operation \( \circ \) on \( H\).
Definition 2.4 [4, 5] Let $H$ be a hyper BCK-algebra. Then, a non-empty subset $I$ of $H$ with $0 \in I$ is called a weak hyper BCK-ideal of $H$ if it satisfies: \((\forall x, y \in H)(x \circ y \subseteq I \text{ and } y \in I \implies x \in I)\); hyper BCK-ideal of $H$ if it satisfies: \((\forall x, y \in H)(x \circ y \ll I \text{ and } y \in I \implies x \in I)\); reflexive hyper BCK-ideal of $H$ if it is a hyper BCK-ideal of $H$ and satisfies: \((\forall x \in H) x \circ x = \{0\}\); strong hyper BCK-ideal of $H$ if it satisfies: \((\forall x, y \in H)(x \circ y \cap I \neq \emptyset \text{ and } y \in I \implies x \in I)\).

Theorem 2.5 [5] Let $S$ be a non-empty subset of a hyper BCK-algebra $H$. Then $S$ is a hyper subalgebra of $H$ if and only if $x \circ y \subseteq S$ for all $x, y \in S$.

Theorem 2.6 [5] Let $H$ be a hyper BCK-algebra. Then the set $S(H) := \{x \in H | x \circ x = \{0\}\}$ is a hyper subalgebra of $H$, which is called BCK-part of $H$.

Theorem 2.7 [5] Every hyper BCK-ideal of a hyper BCK-algebra is a weak hyper BCK-ideal.

Theorem 2.8 [6] Let $A$ be a subset of a hyper BCK-algebra $H$. If $I$ is a hyper BCK-ideal of $H$ such that $A \ll I$, then $A$ is contained in $I$.

Theorem 2.9 [1] Let $\Theta$ be a regular congruence relation on $H$ and $H/\Theta = \{[x]_\Theta | x \in H\}$. Then $H/\Theta$ with hyper operation “$\circ$” and hyper order “$<$” which are defined as follow, is a hyper BCK-algebra which is called quotient hyper BCK-algebra,

\[ [x]_\Theta \circ [y]_\Theta = \{[z]_\Theta : z \in x \circ y\}, \quad [x]_\Theta < [y]_\Theta \iff [0]_\Theta \in [x]_\Theta \circ [y]_\Theta. \]

Theorem 2.10 [1] Let $\Theta$ be a regular congruence relation on $H$. Then

\[ [0]_\Theta \text{ is a reflexive hyper BCK-ideal of } H \iff \frac{H}{\Theta} \text{ is a BCK-algebra.} \]

3 Hyper BCK-algebras with the property R.M.

Definition 3.1 Let $H$ be a hyper BCK-algebra. We say that $H$ satisfies the right multiply property, or is with condition R.M., if the following implication holds:

\[(\forall x, y, z \in H)(x \ll y \implies x \circ z \ll y \circ z).\]
**Example 3.2** (i) Consider a hyper $BCK$-algebra $H = \{0, 1, 2, 3\}$ with the following Cayley’s table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1}</td>
<td>{0}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0}</td>
<td>{2}</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
<td>{1, 3}</td>
<td>{1}</td>
<td>{0, 3}</td>
</tr>
</tbody>
</table>

It is easy to verify that $H$ satisfies condition R.M.

(ii) Consider a hyper $BCK$-algebra $H = \{0, 1, 2\}$ with the following Cayley’s table:

<table>
<thead>
<tr>
<th>$\circ$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0, 1}</td>
<td>{0}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0}</td>
</tr>
</tbody>
</table>

Then $H$ does not satisfy condition R.M., because $1 \preceq 2$ but $\{0, 1\} = 1 \circ 2 \npreceq 2 \circ 2 = \{0\}$

**Lemma 3.3** Let $H$ be a hyper $BCK$-algebra with condition R.M. and $A, B \subseteq H$. If $A \preceq B$, then $A \circ x \preceq B \circ x$ for any $x \in H$.

**Proof.** Let $a \in A$. Since $A \preceq B$, we obtain $a \preceq b$ for some $b \in B$, and so for any $x \in H$, $a \circ x \preceq b \circ x$ by hypothesis. This implies $A \circ x \preceq B \circ x$ for any $x \in H$.

Note that for any hyper $BCK$-algebra $H$ and $A \subseteq H$, if $A$ is a $\preceq$-right scaler of $H$ then $A \subseteq S(H)$ (see [2, Proposition 3.4]), but the converse is not true, in general. The following proposition shows that the converse of the above fact is true whenever the hyper $BCK$-algebra satisfies condition R.M.

**Lemma 3.4** Let $H$ be a hyper $BCK$-algebra with condition R.M. If $A \subseteq S(H)$, then $A$ is a $\preceq$-right scaler of $H$.

**Proof.** Let $a \in A$ and $x \preceq a$ for some $x \in H$. Then $a \in S(H)$, and so $a \circ a = \{0\}$ by Theorem 2.6. On the other hand, by the R.M. property of $H$, $x \preceq a$ implies $x \circ a \preceq a \circ a = \{0\}$. Hence $x \circ a = \{0\}$ by Theorem 2.2(a5). Therefore $A$ is $\preceq$-right scaler.

**Proposition 3.5** Let $H$ be a hyper $BCK$-algebra. Then $S(H)$ is a hyper $BCK$-algebra with condition R.M.
Proof. By Theorem 2.6, $S(H)$ is a hyper $BCK$-algebra. Now, we show that $S(H)$ satisfies condition R.M. Suppose that $x, y \in S(H)$ such that $x \ll y$ and let $z$ be an arbitrary element in $S(H)$. Since $S(H)$ is a $\ll$-right scalar by Lemma 3.4, it follows that $x \circ y = \{0\}$. Hence by Definition 2.1(H1), we have

$$(x \circ z) \circ (y \circ z) \ll x \circ y = \{0\}.$$  

Thus $(x \circ z) \circ (y \circ z) = \{0\}$ and so $x \circ z \ll y \circ z$. Therefore $S(H)$ satisfies condition R.M.

Definition and Lemma 3.6 \cite{3} Let $H$ be a hyper $BCK$-algebra.  

(i) An element $a$ in $H$ is called an atom of $H$ if it satisfies:

$$(\forall x \in H)(x \ll a \implies x = 0 \text{ or } x = a).$$

(ii) A subset $A$ of $H$ is called atomic if each element of $A$ is atom.

If a hyper $BCK$-algebra $H$ is atomic, then

(i) $x \circ y \subseteq \{0, x\}$ for all $x, y \in H$,

(ii) $x \circ y = \{x\}$ for all $x, y \in H$ with $x \neq y$.

Proposition 3.7 Every atomic hyper $BCK$-algebra satisfies condition R.M.

Proof. Let $H$ be an atomic hyper $BCK$-algebra and let $x, y \in H$ be such that $x \ll y$. Then, since $H$ is atomic, $x = 0$ or $x = y$. On the other hand, by Lemma 3.6, we have $x \circ z \subseteq \{0, x\}$ and $y \circ z \subseteq \{0, y\}$ for any $z \in H$. If $x = 0$, then clearly $x \circ z = 0 \circ z = \{0\} \ll y \circ z$ for any $z \in H$. If $x = y$, then $x \circ z = y \circ z$ for any $z \in H$. Therefore $H$ satisfies condition R.M.

The following example shows that the converse of Proposition 3.7 is not true, in general.

Example 3.8 Consider a hyper $BCK$-algebra $H = \{0, 1, 2\}$ with the following Cayley’s table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>${0}$</td>
<td>${0}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>1</td>
<td>${1}$</td>
<td>${0, 1}$</td>
<td>${0}$</td>
</tr>
<tr>
<td>2</td>
<td>${2}$</td>
<td>${2}$</td>
<td>${0, 2}$</td>
</tr>
</tbody>
</table>

It is easy to verify that $H$ satisfies condition R.M. but it is not atomic.

Proposition 3.9 Let $H := H_1 \oplus H_2$ be the union of two hyper $BCK$-algebras $H_1$ and $H_2$. Then $H$ satisfies condition R.M. if and only if $H_1$ and $H_2$ satisfy condition R.M.
Proof. We note that the hyper $BCK$-algebra $(H = H_1 \oplus H_2; \ast, 0)$, the union of two hyper $BCK$-algebras $(H_1; \ast_1, 0)$ and $(H_2; \ast_2, 0)$, is defined as follows: for any $x, y \in H$,

$$x \circ y := \begin{cases} x \circ_1 y & \text{if } x, y \in H_1, \\ x \circ_2 y & \text{if } x, y \in H_2, \\ \{x\} & \text{otherwise} \end{cases}$$

If $H$ satisfies condition R.M., then, since $H_1, H_2 \subseteq H$, they also satisfy the R.M. property.

Conversely, suppose that both $H_1$ and $H_2$ satisfy the R.M. property. Let $x, y \in H$ be such that $x \ll y$. Then, it follows from Definition of $H$ that $x = 0$ or $x, y$ in the same hyper $BCK$-algebra $H_1$ or $H_2$. This implies that, in any case $x \circ z \ll y \circ z$ for all $z \in H$. Therefore $H$ satisfies condition R.M.

**Proposition 3.10** Let $H = H_1 \times H_2$ be the product of two hyper $BCK$-algebras $H_1$ and $H_2$. Then $H$ satisfies condition R.M. if and only if both $H_1$ and $H_2$ satisfy the R.M. property.

**Proof.** Using the Definition of the product of two hyper $BCK$-algebras, the proof is easy.

**Proposition 3.11** Let $f : H \rightarrow K$ be a monomorphism of hyper $BCK$-algebras. If $K$ satisfies condition R.M., then so is $H$.

**Proof.** By the first homomorphism theorem, we have $H \cong f(H)$. Clearly $f(H)$ satisfies condition R.M. Therefore $H$ also satisfies condition R.M.

Now, we consider condition R.M. in the quotient hyper $BCK$-algebras and give a condition for congruence relation $\Theta$ on hyper $BCK$-algebra $H$ such that $H$ and $\frac{H}{\Theta}$ are the same related to condition R.M.

**Definition 3.12** Let $\Theta$ be a congruence relation on hyper $BCK$-algebra $H$. Then $\Theta$ is called strongly regular if whenever $x \circ y \Theta 0$ then $0 \in x \circ y$.

**Lemma 3.13** Every strongly regular relation on a hyper $BCK$-algebra is a regular relation.

**Proof.** Let $\Theta$ be a strongly regular relation on hyper $BCK$-algebra $H$, and let $x, y \in H$ be such that $x \circ y \Theta 0$ and $y \circ x \Theta 0$. Then $0 \in x \circ y$ and $0 \in y \circ x$ and so $x = y$ by the Definition 2.1(a4). Hence $x \Theta y$ by the reflexivity of $\Theta$. Therefore $\Theta$ is a regular relation on $H$.

**Theorem 3.14** Let $\Theta$ be a strongly regular relation on a hyper $BCK$-algebra $H$. Then $H$ satisfies condition R.M. if and only if $\frac{H}{\Theta}$ satisfies condition R.M.
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4 On properties of hyper $BCK$-ideals

It is well known that every hyper $BCK$-ideal is a weak hyper $BCK$-ideal, but the converse is not true in general. The following proposition shows that the converse is true whenever the hyper $BCK$-algebra satisfies condition R.M.

**Proposition 4.1** Assume that $H$ satisfies condition R.M. and $I$ is contained in the $BCK$-part $S(H)$ of $H$. Then $I$ is a weak hyper $BCK$-ideal of $H$ if and only if it is a hyper $BCK$-ideal of $H$.

**Proof.** By Theorem 2.7, we only need to prove the necessity. Assume that $H$ satisfies condition R.M. and $I$ is a weak hyper $BCK$-ideal of $H$. Let $x, y \in H$ be such that $x \circ y \leq I$ and $y \in I$. Then for any $t \in x \circ y$, there is $i \in I$ such that $t \leq i$, and so $t \circ i \leq i \circ i$ by condition R.M. of $H$. Since $i \in S(H)$, we obtain $i \circ i = \{0\}$. Hence $t \circ i \leq \{0\}$ and so $t \circ i = \{0\}$. Thus $t \circ i \subseteq I$. Hence, since $I$ is a weak hyper $BCK$-ideal of $H$ and $i \in I$, we get $t \in I$. Thus $x \circ y \subseteq I$. It follows from $y \in I$ and $I$ is a weak hyper $BCK$-ideal of $H$ that $x \in I$. Therefore $I$ is a hyper $BCK$-ideal of $H$.

**Definition and Theorem 4.2** [5] Let $A$ be a subset of a hyper $BCK$-algebra. The smallest hyper $BCK$-ideal containing $A$ is called the hyper $BCK$-ideal generated by $A$, and is denoted by $(A)$. Then

$$(A) \supseteq \{x \in H|(x \circ a_1) \circ a_2 \circ ... \circ a_n = \{0\} \text{ for some } a_1, a_2, ..., a_n \in A\}.$$
In the following Theorem, we give a characterization of the elements of the hyper $BCK$-ideal generated by $A$.

**Theorem 4.3** Let $H$ be a hyper $BCK$-algebra with condition R.M., and let $A$ be a subset of the $BCK$-part $S(H)$ of $H$ and $x \circ y < \infty$ for all $x, y \in H$. Then

$$(A) = \{x \in H : (\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_n = \{0\}, n \in N \text{ and } a_1, \ldots, a_n \in A\}$$

**Proof.** We denote

$$B := \{x \in H : (\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_n = \{0\}, n \in N \text{ and } a_1, \ldots, a_n \in A\}$$

and prove that $(A) = B$. Since $A \subseteq S(H)$, we have $a \circ a = \{0\}$ for all $a \in A$. This implies that $a \in B$ for all $a \in A$. Hence $A \subseteq B$. Now we show that $B$ is a hyper $BCK$-ideal of $H$. Let $x, y \in H$ be such that $x \circ y \ll B$ and $y \in B$. Since $|x \circ y| < \infty$, we may suppose that $x \circ y = \{t_1, t_2, \ldots, t_m\}$. For every $t_i \in x \circ y$ there exists $z_i \in B$ such that $t_i \ll z_i$. It follows from $z_i \in B$ that $(\ldots((z_i \circ a_1^i) \circ a_2^i)\ldots) \circ a_n^i = \{0\}$ for some $a_1^i, \ldots, a_n^i \in A$. Using condition R.M. of $H$ and $t_i \ll z_i$, we get $t_i \circ a_1^i \ll z_i \circ a_1^i$. Also, by condition R.M. of $H$ and Lemma 3.3, it follows from $t_i \circ a_1^i \ll z_i \circ a_1^i$ that $(t_i \circ a_1^i) \circ a_2^i \ll (z_i \circ a_1^i) \circ a_2^i$. Repeating this way for $a_3^i, \ldots, a_n^i$, we obtain

$$(\ldots((t_i \circ a_1^i) \circ a_2^i)\ldots) \circ a_n^i \ll (\ldots((z_i \circ a_1^i) \circ a_2^i)\ldots) \circ a_n^i = \{0\}.$$ 

Hence $(\ldots((t_i \circ a_1^i) \circ a_2^i)\ldots) \circ a_n^i = \{0\}$. Using Theorem 2.2(a7) and Definition 2.1(H2), we get

$$(\ldots((t_i \circ a_1^i)\ldots) \circ a_1^m)\ldots) \circ a_1^m = \{0\}$$

for all $t_i \in x \circ y$. Thus we have

$$(\ldots(\ldots((x \circ y) \circ a_1^i)\ldots) \circ a_1^m)\ldots) \circ a_1^m \circ a_n^m \circ a_1^m = \{0\}$$

Hence, it follows from $y, a_1^1, \ldots, a_n^m \in B$ that $x \in B$. Therefore $B$ is a hyper $BCK$-ideal of $H$ containing $A$. Since $(A)$ is the smallest hyper $BCK$-ideal containing $A$, we obtain $(A) \subseteq B$. By Theorem 4.2, obviously $B \subseteq (A)$. Therefore $B = (A)$, which completes the proof.

**Definition 4.4** Let $H$ be a hyper $BCK$-algebra and $A \subseteq H$. An element $a \in A$ is called maximum element of $A$ if $x \ll a$ for all $x \in A$.
For any elements $x, y$ of a hyper $BCK$-algebra $H$, we denote
\[ x \circ^n y = (((x \circ y) \circ y) \circ ...) \circ y \]
in which $y$ occurs $n$ times.

**Proposition 4.5** Let $H$ be a hyper $BCK$-algebra with condition $R.M.$, and let $A$ be a subset of $BCK$-part $S(H)$ of $H$. If $a \in A$ is the maximum element of $A$, then
\[ (A) = \{ x \in H : x \circ^n a = \{0\}, \text{ for some } n \in N \}. \]

**Proof.** Let $a \in A$ be the maximum element of $A$. Hence for any $x \in A$, we have $x \ll a$. Using condition $R.M.$ and the fact that $A \subseteq S(H)$, we have $x \circ a \ll a \circ a = \{0\}$, which implies
\[ x \circ a = \{0\}, \text{ for all } x \in A. \tag{1} \]

Now, we denote $B := \{ x \in H : x \circ^n a = \{0\}, \text{ for some } n \in N \}$. Using the Theorem 4.2, we get $B \subseteq (A)$. To prove the converse of inclusion, let $x \in (A)$. Then, it follows from Theorem 4.3 that there exist $a_1, ..., a_n \in A$ such that $\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_n = \{0\}$. This implies $\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_{n-1} \ll a_n$. Using condition $R.M.$ and (1), we get $(\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_{n-1} \ll a_n \circ a = \{0\}$. Hence $\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_{n-1} \ll a_n = \{0\}$, and so $\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_{n-1} = \{0\}$ by Definition 2.1(H2). It follows that $\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_{n-2} \ll a_{n-1}$. Similarly, we can obtain $\ldots((x \circ a_1) \circ a_2)\ldots) \circ a_{n-2} \circ a = \{0\}$, and so
\[ (\ldots((x \circ a_2) \circ a_1)\ldots) \circ a_{n-2} = \{0\}. \]

Repeating this process for $a_{n-2}, ..., a_1$, we get $x \circ^n a = \{0\}$. Hence $x \in B$ by the Definition of $B$. Thus $[A] \subseteq B$. Therefore $[A] = B$, which completes the proof.

**Definition and Lemma 4.6** [2] Let $H$ be a hyper $BCK$-algebra which the its hyper order $\ll$ is transitive. Then $H$ is called a hyper $BCK$-semi lattice if $x \land y := \inf \{x, y\}$ exists and belong to $H$ for any $x, y \in H$. If for $x, y, a \in H$ and $m, n \in N$, $a \circ^m x = \{0\}, a \circ^n y = \{0\}$ and $x \land y$ is a $\ll$-right scalar element of $H$, then there exists a natural number $p$ such that $a \circ^p (x \land y) = \{0\}$.

**Proposition 4.7** Let $H$ be a hyper $BCK$-semi lattice with condition $R.M.$, and let $A, B$ be subsets of the $BCK$-part $S(H)$ of $H$. If $a \in A$ and $b \in B$ are the maximum elements of $A$ and $B$ respectively, then
\[ (A) \cap (B) = \{ x \in H : x \circ^n (a \land b) = \{0\}, \text{ for some } n \in N \}. \]
Proof. Denote by $C$ the set $\{x \in H : x \circ^n (a \land b) = \{0\} \text{ for some } n \in N\}$. It suffices to prove that $(A \cap B) = C$. Since $a \land b \ll a$, it follows from Theorem 2.8 that $a \land b \in (A]$. This implies $C \subseteq (A]$. Similarly, we have $C \subseteq (B]$. Hence $C \subseteq (A] \cap (B]$. To prove the converse of inclusion, let $x \in (A] \cap (B]$. Then, by Theorem 4.5, there exist $s, t \in N$ such that $x \circ^s a = \{0\}$ and $x \circ^t b = \{0\}$. Since $a \land b$ is an element of $BCK$-part, it follows from Lemma 3.4 that $a \land b$ is a $\ll$-right scaler of $H$. Hence, by Lemma 4.6, there exists $p \in N$ such that $x \circ^p (a \land b) = \{0\}$. This implies $x \in C$. Thus $[A] \cap [B] \subseteq C$. Therefore $[A] \cap [B] = C$, which completes the proof.

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