Dade’s Projective Conjecture for $p$-Block with an Extra-Special Defect Group

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Abstract

The aim of this work is to show that Dade’s Projective Conjecture for $p$-block $B$ of a finite group $G$ is equivalent to the equalities between the number of irreducible characters of $B$ with defect $d$ and the number of irreducible characters of Brauer correspondent of $N_{G}((E, b_{E}))$, where $E$ is the defect group of $B$ which is an extra-special $p$-group of order $p^{3}$ and exponent $p$ for an odd prime number $p$ and $(E, b_{E})$ is the maximal $(G, B)$-subpair which is associated to $B$. We shall investigate the case that $O_{p}(G) = Z(E) \leq Z(G)$.

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1 Introduction

Let $p$ be an odd rational prime number. Let $G$ be a finite group such that $|G| = p^{a}m$, with $(p, m) = 1$. Let $B$ be a $p$-block of $G$ with defect group $E$ which is an extra-special $p$-group of order $p^{3}$ and exponent $p$. Local theory with respect to $p$ is to get information of the finite group $G$ and its invariants from the corresponding information of $p$-local subgroups, where we mean by $p$-local subgroups the normalizer of non-trivial $p$-subgroups of $G$. Assume that the unique maximal normal $p$-subgroup of $G$; $O_{p}(G)$ is a central $p$-subgroup of $G$ such that $O_{p}(G) = Z(E)$, where $Z(E)$ is the center of $E$. Let $d$ be an
arbitrary positive integer and $\lambda$ be an irreducible character of $O_p(G)$. Dade’s Projective Conjecture for $B$ asserts that one can express the number of irreducible characters of $B$ of defect $d$ and lying over $\lambda$ as an alternating sum of the corresponding numbers of $p$-local subgroups. For more details, see [1] and [7]. According to G. R. Robinson’s approach, the alternating sum which is mentioned above will be reduced to one which has only chains starting with Alperin-Goldschmidt $(G, B)$-subpairs. See Section 3 below for the definition of Alperin-Goldschmidt $(G, B)$-subpairs. We shall show that Dade’s Projective Conjecture is equivalent to the equalities $k_d(B) = k_d(b)$ for all non-negative integers $d$. Here $b$ is the unique Brauer correspondent of $B$ in $N_G((E,b_E))$ and $k_d(b)$ is the number of irreducible characters with defect $d$ and belong to $b$; see the notation below. This result can be attained if one shows that $(E,b_E)$ is the unique Alperin-Goldschmidt $(G, B)$-subpair, up to conjugation. However, the main idea is to show that the $p$-local subgroup $N_G((E,b_E))$ controls the fusion of $(G, B)$-subpairs, which implies that the contribution from chains starting with $(M,b_M)$ cancel each other out, for each $(G, B)$-subpair $(M,b_M)$, where $M$ is a maximal subgroup of $E$.

2 Notation

We fix $p$ to be a rational odd prime number. $G$ will be a finite group of order $p^a m$ with $g.c.d(p, m) = 1$. The triple $(\mathcal{K}, R, F)$ will be a $p$-modular system. Hence, $R$ is a complete discrete valuation ring with field of fractions equal to $\mathcal{K}$ and residue class field is $F$ which is an algebraically closed field of characteristic $p$. We assume that $R$ contains enough $p$-power roots of unity. We shall use $H \leq G$ to denote that $H$ is a subgroup of $G$. For a subgroup $H$ of $G$, $N_G(H)$ is the normalizer subgroup of $H$ in $G$, while $C_G(H)$ is the centralizer of $H$ in $G$. The center of $H$, the commutator subgroup of $H$, and the Frattini subgroup of $H$ will be denoted by $Z(H)$, $H'$ and $\Phi(H)$ respectively. The unique largest normal $p$-subgroup of $G$ will be denoted by $O_p(G)$. The general linear group of dimension two with entries from the Galois field $GF(p)$ will be denoted by $GL(2, p)$. Let $B$ be a $p$-block of $G$. For a $p$-subgroup $Q$ of $G$, $Br_Q$ will be the Brauer homomorphism from the fixed points of the group algebra $RG$ under the conjugation action by $Q$ to the group algebra $FC_G(Q)$. Any $p$-block of $C_G(Q)$ will be denoted by $b_Q$. Then the pair $(Q,b_Q)$ is called the $(G, B)$-subpair, whenever, $b_Q$ appears in the decomposition of $Br_Q(1_B)$. For a non-negative integer $d$, we say that an irreducible character $\chi$ of $G$ has defect $d$ if the $p$-part of its degree equals $p^a - d$. Then $k_d(B)$ refers to the number of
irreducible characters of $G$ which belong to $B$ and have defect $d$. Furthermore, if $\lambda$ is an irreducible character of a central $p$-subgroup of $G$ then $k_d(B, \lambda)$ denotes the number of irreducible characters of $B$ with defect $d$ and lying over $\lambda$. We have denoted all $(G, B)$-subpairs by $\mathcal{S}(G, B)$. We write $\mathcal{S}(G, B)/G$ for the representatives for the orbits under the action by $G$. We define the chain $\sigma$ to be the strict inclusion of $(G, B)$-subpairs. The number of subpairs which are involved in such $\sigma$ is denoted by $\dim(\sigma)$. The stabilizer of $\sigma$ in $G$ is $N_G(\sigma)$, while $B(\sigma)$ is used to indicate the stabilizer $p$-block of $\sigma$ in $N_G(\sigma)$. It is well known by the work of Knörr and G. R. Robinson in [5] that for each $p$-block $b(\sigma)$ of $N_G(\sigma)$, the induced $p$-block $b(\sigma)^G$ of $G$ is defined. For standard facts of representation theory and group theory, the reader can consult the references [3], [6] and [4].

3 $(G, B)$-subpairs and the cancellation processes

3.1 Introduction

The notion of $(G, B)$-subpairs was introduced in $p$-block theory by Alperin and Broué in [2]. It is a generalization of the concept of $p$-groups. For our purposes, $(G, B)$-subpairs can be used for the cancellation methods which were originated in the work of G. R. Robinson, see [5], [7] and [9]. We shall introduce the notion of $(G, B)$-subpairs and discuss the cancellation methods.

3.2 $(G, B)$-subpairs

Let $G$ be a finite group, $B$ be a $p$-block of $G$. Let $Q$ be an arbitrary $p$-subgroup of $G$. Then, we consider the Brauer map

$$Br_Q : RG^Q \to FC_G(Q)$$

to define the defect group of $B$ to be a maximal $p$-subgroup of $G$ such that $1_B$ does not belong to the kernel of this map. Write $D$ for a defect group of $B$. Then $D$ is a unique $p$-subgroup of $G$ up to conjugacy. It follows that

$$Br_D(1_B) = 1_{b_1} + 1_{b_2} + \cdots + 1_{b_t},$$

where $b_i$ is a $p$-block of $C_G(D)$, for $1 \leq i \leq t$. We say that $b_i$ is in Brauer correspondence with $B$ and we write $b_i^G := B$.

Remarks 3.1  
$\bullet$ By [6, Lemma 3.3, page 321], the defect group of $b_i$ is contained in the defect group of $B$ for $1 \leq i \leq t$. 


In the case that $b_i$ has defect group coinciding with the defect group of $B$, we say that $b_i$ is a root of $B$.

Now for $p$-subgroup $Q$ of $G$, write $b_Q$ to denote a $p$-block of $C_G(Q)$. We call the pair $(Q, b_Q)$ a $(G, B)$-subpair whenever $Q$ is a $p$-subgroup of $G$ and $b_Q$ is a $p$-block of $C_G(Q)$ such that $b_Q^G = B$.

Remarks 3.2

- By [6, Section 9.1 Chapter 5], there is a bijection between $p$-blocks of $C_G(Q)$ and those of $QC_G(Q)$. Hence, we can consider the pair $(Q, b_Q)$, where, $b_Q$ is a $p$-block of $QC_G(Q)$.

- It is clear that the finite group $G$ acts on the set of all $(G, B)$-subpairs. We can define the inclusion of $(G, B)$-subpairs as well as the normal conditions. It turns out that the action of $G$ on the set of all subpairs respects the inclusion. In addition, the maximal $(G, B)$-subpairs are $G$-conjugated. See [6, Theorem 9.1, Chapter 5].

- Let $(Q, b_Q)$ be an arbitrary $(G, B)$-subpair. Then there is a maximal $(G, B)$-subpair $(D, b_D)$ such that $(Q, b_Q) \leq (D, b_D)$. Furthermore, if $(P, b_P)$ is a $(G, B)$-subpair and $Q$ is a subgroup of $P$ then there exists one and only one $p$-block $b_Q$ of $C_G(Q)$ such that $(Q, b_Q) \leq (P, b_P)$. See [6, Theorem 9.3, Chapter 5].

- If $B$ is the principal $p$-block of $G$ then $(G, B)$-subpairs correspond canonically to $p$-subgroups of $G$.

Now we define an Alperin-Goldschmidt $(G, B)$-subpair as follows: For a $p$-block $B$ of $G$ with defect group $D$. Let $(U, b_U)$ be an arbitrary $(G, B)$-subpair. We say that $(U, b_U)$ is an Alperin-Goldschmidt $(G, B)$-subpair if we have

1. $C_D(U) = Z(U)$.
2. $O_p(N_G(\langle U, b_U \rangle)/UC_G(U)) = 1$.
3. $N_D(U)$ is a defect group of $1_{b_U} \cdot RN_G(\langle U, b_U \rangle)$.

3.3 The cancellation methods

Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $D$. Denote the totality of all $(G, B)$-subpairs which is contained in the maximal subpair
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(D, b_D) by S(G, B). According to the inclusion of (G, B)-subpairs, we define the chain of (G, B)-subpairs as follows:

$$\sigma : (Q_1, b_{Q_1}) < (Q_2, b_{Q_2}) < \ldots < (Q_n, b_{Q_n}).$$

It is clear that G acts on S(G, B). We shall write the representatives of the orbits under this action by S(G, B)/G. The stabilizer of $\sigma$ is the finite group

$$N_G(\sigma) := \cap_{1 \leq i \leq n} N_G((Q_i, b_{Q_i})).$$

The dimension of $\sigma$ is $\text{dim}(\sigma) = n$ and the stabilizer $p$-block of $\sigma$ is

$$B(\sigma) := Br_{Q_n}(1_B) FN_G(\sigma).$$

We will denote the initial subpair by $(Q_\sigma, b_\sigma)$ and the final subpair by $(Q^\sigma, b^\sigma)$. By [5], for each $p$-block $b(\sigma)$ which is a direct summand of $B(\sigma)$, $b(\sigma)^G$ is defined and equals to $B$. Also, the stabilizer of $\sigma$ is the inertial group of $b^\sigma$ in the intersection of the normalizer of all $p$-groups which is involved in $\sigma$. We shall consider the subset of S(G, B) in such a way that each member is an Alperin-Goldschmidt (G, B)-subpair.

Consider the function

$$f : S(G, B) \to A$$

where A is an abelian group. This function depends only on the chain stabilizer and the initial (G, B)-subpair of an arbitrary element of S(G, B). The following Lemma is analogous to that in [8, Corollary 1.2].

**Lemma 3.3** With the notation above, let $(Q, b_Q)$ be an arbitrary (G, B)- subpair which is not an Alperin-Goldschmidt (G, B)-subpair. Then chains which start with $(Q, b_Q)$ have zero contribution to the alternating sum

$$\sum_{\sigma \in S(G, B)/G} (-1)^{\text{dim}(\sigma)} f(\sigma).$$

4. A certain action of the inertial quotient on a maximal subgroup of an extra special $p$-group

4.1 Introduction

In this section, we shall concern with the action of the inertial quotient of an arbitrary (G, B)-subpair on an arbitrary maximal subgroup of an extra special $p$-group of order $p^3$ and exponent $p$, for an odd prime number $p$. 
A finite $p$-group $E$ is called an extra-special $p$-group if

$$Z(E) = \Phi(E) = E' \cong C_p.$$ 

Here $C_p$ means a cyclic group of order $p$. We have seen in the book [4] that there are two types of extra-special $p$-groups, up to isomorphism. This dichotomy is according to the exponent. We shall consider an extra-special $p$-group of order $p^3$ and exponent $p$. We shall denote this group by $E$. From now on, we write the extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$ in the form $E := \langle x, y, z : x^p = y^p = z^p = [x, z] = [y, z] = 1; [x, y] = z \rangle$. We choose an arbitrary maximal subgroup $M = \langle x, z \rangle$ of $E$. We assume that $O_p(G) = Z(E) = \langle z \rangle \leq Z(G)$. Therefore, $M = O_p(G) \times \langle x \rangle$.

Now Let $b_M$ be a $p$-block of $C_G(M)$. Then $(M, b_M)$ is a $(G, B)$-subpair. The $p$-local subgroup associated with $(M, b_M)$ is the subgroup

$$N_G((M, b_M)) = \{ g \in G | M^g = M; b_M^g = b_M \}.$$ 

We shall call $N_G((M, b_M))/C_G(M)$ the inertial quotient and studying its action on $M$. On the other hand, since $M$ is an elementary abelian $p$-group of order $p^2$, we can regard it as a vector space over the Galois field $GF(p)$ with the following basis: $z := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $x := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now the conjugation action embeds the inertial quotient $N_G((M, b_M))/C_G(M)$ into $GL(2, p)$.

**Lemma 4.1** Let $M$ be a maximal subgroup of an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$. Assume that $M = O_p(G) \times \langle x \rangle$ and $O_p(G) \leq Z(G)$. Then the image of $N_G((M, b_M))/C_G(M)$ in $GL(2, p)$ consists of upper triangular matrices.

**Proof:** Let $\bar{n} := nC_G(M)$ be an arbitrary element in $N_G((M, b_M))/C_G(M)$. Since $GL(2, p)$ is the Automorphism group of $M$, $\bar{n}$ can be identified with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $ad - bc \neq 0$. Since $O_p(G) \leq Z(G)$ and $1 < O_p(G) < M$, we have $\bar{n}^z = z$. It follows that $a = 1$ and $c = 0$. Therefore, the image of $N_G((M, b_M))/C_G(M)$ in $GL(2, p)$ consists of upper triangular matrices.

**Corollary 4.2** With the notation above, $N_G((M, b_M))/C_G(M)$ has a normal Sylow $p$-subgroup of order $p$ and $EC_G(M)$ is a normal subgroup of $N_G((M, b_M))$.

**Proof:** By Lemma 4.1, $N_G((M, b_M))/C_G(M)$ consists of upper triangular matrices. In fact, $\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in GF(p) \}$ is the unique Sylow $p$-subgroup
of $N_G((M, b_M))/C_G(M)$. Now $(M, b_M)$ is a normal $(G, B)$-subpair of $(E, b_E)$. In particular, $E \leq N_G(M, b_M)$. Therefore, $EC_G(M)/C_G(M) \cong E/M$ is the unique Sylow $p$-subgroup of $N_G((M, b_M))/C_G(M)$ and clearly, $EC_G(M)$ is a normal subgroup of $N_G(M, b_M)$.

**Theorem 4.3** Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $E$ which is an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$. Assume that $O_p(G) = Z(E) \leq Z(G)$. Then $(M, b_M)$ is not Alperin-Goldschmidt $(G, B)$-subpair for each maximal subgroup $M$ of $E$.

Proof: Since $O_p(N_G((M, b_M))/C_G(M)) \neq 1$, for each maximal subgroup $M$ of $E$, the result follows.

**Theorem 4.4** Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $E$ which is an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$. Assume that $O_p(G) = Z(E) \leq Z(G)$. Then $(E, b_E)$ is the unique Alperin-Goldschmidt $(G, B)$-subpair, up to conjugation.

Proof: That $(E, b_E)$ is an Alperin-Goldschmidt $(G, B)$-subpair is clear. However, the uniqueness is guaranteed by Theorem 4.3. Therefore, we get the following theorem which measures the fusion in the case that $O_p(G)$ is a central $p$-subgroup of $G$ which equals the center of $E$.

**Theorem 4.5** Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $E$ which is an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$. Assume that $O_p(G) = Z(E) \leq Z(G)$. Then $N_G((E, b_E))$ controls fusion of $(G, B)$-subpairs.

Proof: It suffices to show that

$$N_G((M, b_M)) = (N_G((E, b_E)) \cap N_G((M, b_M)))C_G(M),$$

for each maximal subgroup $M$ of $E$. It is clear that the right hand side is contained in $N_G((M, b_M))$. Since $EC_G(M)$ is a normal subgroup of $N_G((M, b_M))$, we have $N_G(M, b_M) = N_G(M, b_M) \cap N_G(EC_G(M))$. However, $(M, b_M)$ is a normal $(G, B)$-subpair of $(E, b_E)$. Hence, $N_G(EC_G(M)) = N_G(E, b_E)C_G(M)$. It follows, that $N_G((M, b_M)) = (N_G((E, b_E)) \cap N_G((M, b_M)))C_G(M)$. The proof is complete.

**Remarks 4.6** Theorem 4.5 says that the action of $N_G(M, b_M)$ on certain objects is the same action as one given by $N_G(M, b_M) \cap N_G(E, b_E)$. 
5 The prediction of the conjectures for $p$-block with an extra-special defect group

5.1 Introduction

The conjectures that we are tackling with are descended from Alperin and McKay conjectures. In this section, we shall state Dade’s Projective Conjecture. Then we shall show that for a $p$-block $B$ with defect group $E$ which is an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$, this conjecture predicts the equality between the number of irreducible characters of $B$ with defect $d$ and the corresponding number of Brauer correspondent with $B$ in $N_G((E, b_E))$.

5.2 The conjecture under consideration

Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $D$. Assume that $|G| = p^a m$, with $(p, m) = 1$. For an arbitrary non-negative integer $d$, denote the number of irreducible characters of $G$ which belong to $B$ and the $p$-part of their degrees are $p^{a-d}$ by $k_d(B)$. We state Dade’s Projective Conjecture and we offer some remarks.

**Dade’s Projective Conjecture:** Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group which is not central. Assume that $O_p(G)$ is central $p$-subgroup of $G$. Let $\lambda$ be an arbitrary irreducible character of $O_p(G)$. Dade’s Projective Conjecture asserts that for any positive integer $d$, we should have

$$k_d(B, \lambda) = \sum_{\sigma \in S(G,B)/G} (-1)^{\dim(\sigma)} k_d(B(\sigma), \lambda)$$

**Remarks 5.1**

1. It is well known that Dade’s Projective Conjecture is equivalent, after we sum over all irreducible characters of $O_p(G)$, to

$$k_d(B) = \sum_{\sigma \in S(G,B)/G} (-1)^{\dim(\sigma)} k_d(B(\sigma)),$$

for each positive integer $d$.

2. It follows from [7, Theorem 5.1], that the only chains up to $G$-conjugacy which contribute to any alternating sum of a minimal counterexample of Dade’s Projective Conjecture are those whose initial subgroups $Q_\sigma$ satisfies $C_D(Q_\sigma) \subseteq Q_\sigma$. 
5.3 The Main Results

Let $B$ be a $p$-block of a finite group $G$. Assume that $B$ has defect group isomorphic to an extra-special $p$-group $E$ of order $p^3$ and exponent $p$ for an odd prime number $p$. Write $E := \langle x, y, z : x^p = y^p = z^p = [x, z] = [y, z] = 1; [x, y] = z \rangle$. Let $M$ be a maximal subgroup of $E$ such that $M = O_p(G) \times \langle x \rangle$ and $O_p(G) \leq Z(G)$. Now Remark 5.1, (2) implies that the initial subgroup of any chain which has proper contribution to the alternating sum of Dade Projective Conjecture cannot be a cyclic group of order $p$. As a result of this and using Lemma 3.3 and Theorem 4.3, we have the following:

**Theorem 5.2** Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $E$ which is an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$. Assume that $O_p(G)$ is a central subgroup of $G$ which is the center of the defect group of $B$. Denote the chains which start with the $(G, B)$-subpair $(M, b_M)$ by $S(G, B)((M, b_M))$, then, for each positive integer $d$, we have

$$0 = \sum_{\sigma \in S(G, B)((M, b_M)) \cap G} (-1)^{\dim(\sigma)} k_d(B(\sigma))$$

Finally, we are ready to state the main result.

**Theorem 5.3** Let $G$ be a finite group, $B$ be a $p$-block of $G$ with defect group $E$ which is an extra-special $p$-group of order $p^3$ and exponent $p$ for an odd prime number $p$. Assume that $O_p(G)$ is a central subgroup of $G$ which is the center of the defect group of $B$. Then, Dade’s Projective Conjecture holds if, and only if, for each positive integer $d$, $k_d(B) = k_d(b)$, where $b$ is the unique $p$-block of $N_G((E, b_E))$ which is the Brauer correspondent of $B$ and covers $b_E$.

Proof: This is a consequence of Theorem 5.2 and Theorem 4.4.

References


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