The Multiplicity of Some Blow-up Algebras of Equimultiple Ideals

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Abstract

Let $I$ be an equimultiple ideal of Noetherian local ring $A$. This paper gives some multiplicity formulas for some blow-up algebras of $I$. As an application, we obtain some criteria for the Cohen-Macaulayness of extended Rees algebras.

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1 Introduction

Throughout this paper, let $(A, m)$ be a Noetherian local ring of Krull dimension $d > 0$ with infinite residue field $k = A/m$. Let $I$ be an ideal of $A$. Then one calls the blow-up algebras $R(I) = \bigoplus_{n \geq 0} I^nt^n$; $G(I) = \bigoplus_{n \geq 0} (I^n/I^{n+1})t^n$; $T = A[t^{-1}, It]$ the Rees algebra of $I$; the associated graded ring; the extended Rees algebra, respectively. These algebras are important objects of Commutative Algebra and Algebraic Geometry. Moreover, which are closely related by virtue of the isomorphisms $G(I) \cong R(I)/IR(I)$ and $T/t^{-1}T \cong G(I)$.

The multiplicity of blow-up algebras is usually an interesting problem. This problem is concerned by many authors in the past years. Several of authors expressed the multiplicity of some blow-up algebras in terms of mixed multiplicities, e.g. Verma in [16] for Rees algebras; Katz and Verma in [8] for extended Rees algebras. Inevitably, one wants to transmute mixed multiplicities into the Hilbert-Samuel multiplicity. This transmutation was solved by Risler and Teissier [13] in 1973 for two $m$-primary ideals; Rees [10] in 1981 for a set of $m$-primary ideals; Viet [17] in 2000 for a set of arbitrary ideals. By using the results of [16]; [8]; [17]; [18]; [19]; [20], this paper will give some formulas for
mixed multiplicities and equimultiple ideals (Section 3). As an
application of Section 3, we obtain some criteria for the Cohen-Macaulayness
of extended Rees algebras (Section 4), and Cohen-Macaulay extended Rees
algebras with the minimal multiplicity (Section 5).

This paper is divided into five sections. Section 2 deals with some basic
facts on mixed multiplicities and (FC)-sequences of ideals. Section 3 is devoted
to the discussion of mixed multiplicities and the multiplicity of blow-up alge-
bras. Section 4 deals with the Cohen-Macaulayness of extended Rees algebras.
Section 5 gives some results on Cohen-Macaulay extended Rees algebras with
the minimal multiplicity.

2 Mixed multiplicities and equimultiple ideals

This section introduces some concepts and previous important results which
will be used in the paper.

An ideal $J$ is called a reduction of ideal $I$ if $J \subseteq I$ and $I^{n+1} = JJ^n$ for some
$n$. A reduction $J$ is called a minimal reduction of $I$ if it does not properly
contain any other reduction of $I$ [9]. The least integer $n$ such that $I^{n+1} = JJ^n$
is called the reduction number of $I$ with respect to $J$ and we denote it by $r_J(I)$. The reduction number of $I$
is defined by

$$r(I) = \min \{ r_J(I) \mid J \text{ is a minimal reduction of } I \}.$$ 

Set $\ell(I) = \dim \bigoplus_{n \geq 0} (I^n/mI^n)$. Then one called $\ell(I)$ the analytic spread of $I$. If
the residue field $k = A/m$ is infinite, then the minimal number of generators
of every minimal reduction of $I$ is equal to $\ell(I)$ [9]. It is well known that

$$\text{ht } I \leq \ell(I) \leq \dim A,$$

and $I$ is called an equimultiple ideal if $\ell(I) = \text{ht } I$.

Let $J$ be an $m$-primary ideal and $I_1, I_2, \ldots, I_s$ ideals of $A$ such that $J = I_1 \cdots I_s$ is non nilpotent. Suppose that $\dim A/0 : \mathfrak{J}^\infty = q$. Then the Hilbert-
Samuel function $B(n, n_1, \ldots, n_s) = l_A(J^{n_1}I_1^{n_1} \cdots I_s^{n_s})$ is a polynomial of degree $q - 1$ for all large values of $n, n_1, \ldots, n_s$. This polynomial is called the Bhat-
tacharya polynomial [3] of $(J, I_1, \ldots, I_s)$. The terms of total degree $q - 1$ in this
degree of the joint reduction $[10].$
The question arises as to what happens for mixed multiplicities of arbitrary ideals. In answer to this question, Viet (2000) [17] built a sequence of elements called an \((FC)\)-sequence. In this paper, we use just to weak-(FC)-sequences of two ideals. The notion of those sequences is recalled as follows: Set

\[ a : b^\infty = \{ x \in A \mid \text{there is a positive integer } n \text{ such that } xb^n \subseteq a \}. \]

**Definition ([17]).** Let \( J \) be \( m \)-primary and \( I \) a non-nilpotent ideal of \( A \). Set \( A^* = A/0 : I^\infty; \ J^* = JA^*; I^* = IA^* \). Recall that an element \( x \in I \) (\( x^* \) is the image of \( x \) in \( A^* \)) is called a weak-(FC)-element with respect to \((J, I)\) if there exists an integer \( n_1 \) such that

\[ \text{(FC1): } x \in I \setminus mI \text{ and for all } n \geq n_1 \text{ and for all non-negative integers } m, \]

\[ J^mI^{n+1} \cap (x^*) = J^mI^{n+1}x^* . \]

**Remark 2.1.** Let \( J \) be an equimultiple ideal with \( \text{ht}I = h \geq 0 \) and \( J \) \( m \)-primary ideal. When if \( x_1, x_2, \ldots, x_h \) is a weak-(FC)-sequence in \( I \) with respect to \((J, I)\), then \( x_1, x_2, \ldots, x_h \) is a filter-regular sequence with respect to \( I \) and \( x_1, x_2, \ldots, x_h \) is also the part of a system of parameter. Hence there exist \( x_{h+1}, \ldots, x_d \) such that \( q = (x_1, x_2, \ldots, x_h, x_{h+1}, \ldots, x_d) \) is a parameter ideal of \( I \). By [2], we have \( e(q; A) = e(x_i, \ldots, x_d; A/(x_1, \ldots, x_{i-1})) \) for all \( i \leq h \).
The results of [17], [18] proved that (FC)-sequences carry important information on mixed multiplicities. One of interesting results is the following theorem which will be used in this paper.

**Lemma 2.2 ([18]).** Let \( \mathfrak{J} \) be \( \mathfrak{m} \)-primary and \( I \) an equimultiple ideal of \( A \). Suppose that \( x_1, x_2, \ldots, x_h \) is a weak-(FC)-sequence in \( I \) with respect to \( (\mathfrak{J}, I) \). \( R(I) = A[It] \) is the Rees algebra of \( I \). Then

(i) \( e(\mathfrak{J}^{[d-i]}, I^{[i]}) = 0 \) for all \( i \geq h \).

(ii) \( e(\mathfrak{J}^{[d-i]}, I^{[i]}) = e(\mathfrak{J}; A/(x_1, x_2, \ldots, x_i)) \) for all \( i \leq h - 1 \).

(iii) \( e((\mathfrak{J}, It)R(I)) = \sum_{i=0}^{h-1} e(\mathfrak{J}; A/(x_1, x_2, \ldots, x_i)) \).

**Remark 2.3.** Let \( I \) be an equimultiple ideal with \( \text{ht} I = h > 0 \). Then there exists \( I' = (x_1, \ldots, x_h) \) is a minimal reduction of \( I \) and \( \text{ht} I' = \text{ht} I = h \). Thus \( I' \) is an ideal of the principal class. Hence there exists \( J = (y_1, \ldots, y_{d-h}) \) such that \( (x_1, \ldots, x_h, y_1, \ldots, y_{d-h}) \) is a parameter ideal. Therefore, we always choose an ideal \( J = (y_1, \ldots, y_{d-h}) \) such that \( (I + J) \) is \( \mathfrak{m} \)-primary. Set \( A_i = A/(x_1, \ldots, x_i) \) for all \( i \leq h - 1 \). By Lemma 2.2, we have \( e((J^m + I)^{[d-i]}, I^{[i]}) = e((J^m + I), A_i) \). Note that \( (y_1^m, \ldots, y_{d-h}^m, x_1, \ldots, x_h) \) is a reduction of \( J^m + I \). This gives \( e((J^m + I), A_i) = e((y_1^m, \ldots, y_{d-h}^m, x_1, \ldots, x_h), A_i) \). It is well known that \( e((y_1^m, \ldots, y_{d-h}^m, x_1, \ldots, x_h), A_i) = m^{d-h} e((y_1, \ldots, y_{d-h}, x_1, \ldots, x_h), A_i) \).

By Lemma 2.2, \( e((y_1, \ldots, y_{d-h}, x_1, \ldots, x_h), A_i) = e((J + I)^{[d-i]}, I^{[i]}) \). Hence \( e((J^m + I)^{[d-i]}, I^{[i]}) = m^{d-h} e((J + I)^{[d-i]}, I^{[i]}) \) for all \( i \leq h - 1 \).

This property of equimultiple ideals will be used in the next sections.

## 3 The multiplicity of blow-up algebras

In this section, we will determine some multiplicity formulas of blow-up algebras of equimultiple ideals. One of results is the following theorem.

**Theorem 3.1.** Let \( (A, \mathfrak{m}) \) be a Noetherian local ring of dimension \( d > 0 \) with maximal ideal \( \mathfrak{m} \) and infinite residue field \( k = A/\mathfrak{m} \). Let \( I \) be an equimultiple ideal with \( \text{ht} I = h > 0 \). Suppose that \( J = (y_1, y_2, \ldots, y_{d-h}) \) be an ideal of \( A \) such that \( \mathfrak{J} = (J + I) \) is an \( \mathfrak{m} \)-primary ideal. \( T = A[It, t^{-1}] \) is the extended Rees algebra of \( I \), \( R(I) = A[It] \) is the Rees algebra of \( I \). Then

(i) \( e((\mathfrak{J}^m + I)^{[d-i]}, I^{[i]}) = m^{d-h} e(\mathfrak{J}) \) for all \( i \leq h - 1 \) and \( m > 0 \).

(ii) \( e((\mathfrak{J}, t^{-1}, It)T) = e(\mathfrak{J}) \).

(iii) \( e((\mathfrak{J}, It)R(I)) = h.e(\mathfrak{J}) \).

**Proof.** Suppose that \( J' \) is a minimal reduction of \( I \). By Note 1, \( J' \) is generated by a maximal weak-(FC)-sequence \( x_1, x_2, \ldots, x_h \) in \( I \) with respect to \( (\mathfrak{J}, I) \).
Note that $\mathfrak{J} \subseteq \sqrt{(J' + J)}$, then $x_1, x_2, \ldots, x_h, y_1, y_2, \ldots, y_{d-h}$ is a system of parameters of $A$ and is a reduction of $\mathfrak{J}$. Since $\mathfrak{J}^m + I = J^m + I$,
\[ e((\mathfrak{J}^m + I)^{[d-i]}, I^{[i]}) = e((J^m + I)^{[d-i]}, I^{[i]}). \]

On the other hand by Remark 2.3, we have
\[ e((J^m + I)^{[d-i]}, I^{[i]}) = m^{d-h}e((J + I)^{[d-i]}, I^{[i]}) \]
for all $i < h$. Consequently, $e((\mathfrak{J}^m + I)^{[d-i]}, I^{[i]}) = m^{d-h}e(\mathfrak{J}^{[d-i]}, I^{[i]})$ for all $i < h$. Now, by Lemma 2.2 we get $e(\mathfrak{J}^{[d-i]}, I^{[i]}) = e(\mathfrak{J}; A/(x_1, \ldots, x_i))$ for all $i < h$. Since $(x_{i+1}, \ldots, x_h, J)[A/(x_1, \ldots, x_i)]$ is a reduction of $\mathfrak{J}[A/(x_1, \ldots, x_i)]$, $e(\mathfrak{J}; A/(x_1, \ldots, x_i)) = e((x_{i+1}, \ldots, x_h, J); A/(x_1, \ldots, x_i))$. By the above facts, we obtain $e(\mathfrak{J}^{[d-i]}, I^{[i]}) = e((x_{i+1}, \ldots, x_h, J); A/(x_1, \ldots, x_i))$ for all $i < h$. Since $i < h = h \text{t}_I$ and $x_1, \ldots, x_h$ is a weak-(FC)-sequence in $I$ with respect to $(\mathfrak{J}, I)$,
\[ e((x_{i+1}, \ldots, x_h, J); A/(x_1, \ldots, x_i)) = e(J' + J; A) \]
by Remark 2.1. Note that $J' + J$ is a minimal reduction of $\mathfrak{J}$, $e(J' + J) = e(\mathfrak{J})$. Thus $e(\mathfrak{J}^{[d-i]}, I^{[i]}) = e(\mathfrak{J})$.

Hence we get (i): $e((\mathfrak{J}^m + I)^{[d-i]}, I^{[i]}) = m^{d-h}e(\mathfrak{J})$ for all $i \leq h - 1$. We obtained the following facts:
\[ e((\mathfrak{J}, t^{-1}, It)T) = 2^{-d}[e(\mathfrak{J}^2 + I) + \sum_{i=0}^{d-1} \sum_{i=0}^{d-1} 2^i e((\mathfrak{J}^2 + I)^{[d-i]}, I^{[i]})] \]
by [8]; $e((\mathfrak{J}^2 + I)^{[d-i]}, I^{[i]}) = 0$ if and only if $i \geq h$ by Lemma 2.2(i); and
\[ e((\mathfrak{J}^2 + I)^{[d-i]}, I^{[i]}) = 2^{d-h}e(\mathfrak{J}) \]
for all $i \leq h - 1$ by (i). Moreover, note that $(x_1, x_2, \ldots, x_h, y_1, y_2, \ldots, y_{d-h})$ is a reduction of $\mathfrak{J}$ and $(x_1, x_2, \ldots, x_h, y_1^2, y_2^2, \ldots, y_{d-h}^2)$ is also a reduction of $(\mathfrak{J}^2 + I) = (J^2 + I)$. Hence
\[ e(\mathfrak{J}^2 + I) = e((x_1, x_2, \ldots, x_h, y_1^2, y_2^2, \ldots, y_{d-h}^2)) = 2^{d-h}e((x_1, x_2, \ldots, x_h, y_1, y_2, \ldots, y_{d-h})) = 2^{d-h}e(\mathfrak{J}). \]

The above facts imply that $e((\mathfrak{J}, t^{-1}, It)T) = 2^{-h}[1 + \sum_{i=0}^{h-1} 2^i]e(\mathfrak{J}) = e(\mathfrak{J})$. We get (ii). Since $I$ is an equimultiple ideal with $ht I = h > 0$, by [16] and [18] we have $e((\mathfrak{J}, It)R(I)) = \sum_{i=0}^{h-1} e(\mathfrak{J}^{[d-i]}, I^{[i]}).$ Hence by (i) we get $e((\mathfrak{J}, It)R(I)) = \sum_{i=0}^{h-1} e(\mathfrak{J}^{[d-i]}, I^{[i]}).$ Hence by (i) we get $e((\mathfrak{J}, It)R(I)) = h.e(\mathfrak{J}).$

Assume that $(x_1, x_2, \ldots, x_h)$ is a minimal reduction of an equimultiple ideal $I$, and $x_{h+1}, \ldots, x_d$ are elements of $A$ such that $x_1, x_2, \ldots, x_h, x_{h+1}, \ldots, x_d$ is a system of parameter of $A$. We shall denote by $a^*$ the initial form of an element $a$ of $T$ in $G(I); N$ and $M$ the maximal homogeneous ideals of $T$ and $G(I)$, respectively. Then it is easily seen that $x_1 t, x_2 t, \ldots, x_h t, x_{h+1}, \ldots, x_d, t^{-1}$ and
Let $I$ be an equimultiple ideal of $A$ with $\text{ht } I = h > 0$ and $(x_1, x_2, \ldots, x_h)$ a minimal reduction of $I$. Suppose that $x_{h+1}, \ldots, x_d$ are elements of $A$ such that $x_1, x_2, \ldots, x_h, x_{h+1}, \ldots, x_d$ is a system of parameter of $A$. Then we have

(i) $e((I + I', It, t^{-1}); T) = e(Q; T) = e(Q^*; G(I)) = e((I'', I^*t); G(I)) = e(q; A)$.

(ii) $e(G(I)) = e(q; A)$ if $(x_{h+1}, \ldots, x_d)(A/I)$ is a minimal reduction of $\mathfrak{m}(A/I)$.

(iii) $e((I + I', It); R(I)) = e(P; R(I)) = h.e(q; A)$.

Let $S = \bigoplus_{n \geq 0} S_n$ be a graded algebra generated by finitely many elements of degree 1 over an infinite field $k$, $\mathfrak{n}$ the maximal graded ideal of $S$, and $\dim S = d > 0$. For any homogeneous ideal $I$ of $S$, it is well-known that if $c_1 \leq c_2 \leq \cdots \leq c_t$ are the degrees of the elements of an arbitrary homogeneous minimal basis of $I$ then the sequence $c_1, \ldots, c_t$ does not depend on the choice of the minimal basis, and this sequence is called the degree sequence of $I$.

Let $I$ be a homogeneous ideal of $S$ with $\text{ht } I = h > 0$. An ideal $J$ is called a homogeneous minimal reduction of $I$ if $J$ is a minimal reduction of $I$ and $J$ is homogeneous. $I$ is called a homogeneous equimultiple ideal if there exists a homogeneous minimal reduction $J$ of $I$ generated by $h$ homogeneous elements.

When $I$ is a homogeneous ideal of the principal class of $S$ and $c_1, c_2, \ldots, c_h$ is the degree sequence of $I$, upon special computations, Trung in [14] proved that $e(G(I)) = c_1c_2 \cdots c_h e(S)$. Now assume that $q$ is a homogeneous ideal of the principal class and $\mathfrak{n}$-primary of $S$ with $c_1, c_2, \ldots, c_d$ is the degree sequence of $q$. Then since $e(G(q)) = e(q; S)$ and by Trung [op.cit.], $e(G(q)) = c_1c_2 \cdots c_d e(S)$. Hence $e(q; S) = c_1c_2 \cdots c_d e(S)$. Consequently, we have the following result.
Lemma 3.3. Let $q$ be a homogeneous ideal of the principal class and $n$-primary of $S$. Suppose that $c_1, c_2, \ldots, c_d$ is the degree sequence of $q$. Then

$$e(q; S) = c_1c_2 \cdots c_d e(S).$$

Now we give a simple proof of this result.

Proof. Set $c = c_1c_2 \cdots c_d$ and $b_i = c/c_i$ for all $1 \leq i \leq d$. Suppose $q = (x_1, x_2, \ldots, x_d)$ with $\deg x_i = c_i$ for all $1 \leq i \leq d$. Then we have $\deg x_i^b = c$. So $q^* = (x_1^{b_1}, x_2^{b_2}, \ldots, x_d^{b_d}) \subseteq S$. From the above facts we get $b_1b_2 \cdots b_de(q; S) = e(q^*; S) = c^de(S)$. Hence $e(q; S) = c_1c_2 \cdots c_de(S)$. \hfill \Box

As a direct corollary of Corollary 3.2 and Lemma 3.3, we have the following.

Corollary 3.4. Let $c_1, \ldots, c_h$ be the degree sequence of a homogeneous minimal reduction of a homogeneous equimultiple ideal $I$ with $ht I = h > 0$. Suppose that $x_1, \ldots, x_{d-h}$ are homogeneous elements of $S$ such that $(x_1, \ldots, x_{d-h}, I)$ is $n$-primary. Set $I' = (x_1, x_2, \ldots, x_{d-h})$ and $\deg x_i = v_i$ for all $1 \leq i \leq d-h$. Then

(i) $e((I + I', It, t^{-1}); T) = e((I', It); G(I)) = c_1c_2 \cdots c_h v_1 \cdots v_{d-h} e(S)$.

(ii) $e(G(I)) = c_1c_2 \cdots c_h e(S)$.

(iii) $e((I + I', It); R(I)) = hc_1c_2 \cdots c_h v_1 \cdots v_{d-h} e(S)$.

Proof. By Corollary 3.2 and Lemma 3.3 we immediately get (i) and (iii). Now we have only to prove (ii). Suppose that $(y_1, y_2, \ldots, y_h)$ is a homogeneous minimal reduction of $I$. Set $q = (y_1, y_2, \ldots, y_h, x_1, x_2, \ldots, x_{d-h})$. By Corollary 3.2(ii), $e(G(I)) = e(q)$ when $I'(S/I)$ is a minimal reduction of $n(S/I)$. Note that if $I'(S/I)$ is a minimal reduction of $n(S/I)$, then $\deg x_i = v_i = 1$ for all $1 \leq i \leq d-h$. By Lemma 3.3, $e(q) = c_1c_2 \cdots c_h v_1 \cdots v_{d-h} e(S) = c_1c_2 \cdots c_h e(S)$. Hence we get (ii). \hfill \Box

4 The Cohen-Macaulay extended Rees algebras

This section we will discuss the Cohen-Macaulayness of the extended Rees algebra $T = A[It, t^{-1}]$ and the associated graded ring $G(I)$ of an equimultiple ideal $I$. Recall that $T/t^{-1}T \cong G(I)$ and $t^{-1}$ is a regular element in $T$. Consequently, $T$ is Cohen-Macaulay if and only if $G(I)$ is Cohen-Macaulay.

Let $(A, \mathfrak{m})$ be a Cohen-Macaulay ring and $I$ an $\mathfrak{m}$-primary ideal. Hochster and Ratliff showed in [7] that if $I$ is a parameter ideal then $T$ is Cohen-Macaulay. Sally [11] proved that if $A$ has minimal multiplicity then the associated graded ring of $\mathfrak{m}$ is Cohen-Macaulay. Valla generalized these results in [15] as follows: If $I$ has the reduction number $r(I) \leq 1$, then $G(I)$ is Cohen-Macaulay. In [12], Sally showed that $G(\mathfrak{m})$ is Cohen-Macaulay if $r(\mathfrak{m}) \leq 2$. To
By Theorem 3.1, we have

\[ (i) \begin{array}{l}
\text{Cohen-Macaulay and the following conditions are satisfied:}
\end{array} \]

Thus, of Theorem 5.1 [20], we have an ideal of \( B \) that
\[ \text{Cohen-Macaulay if and only if } \]
\[ \text{recover the above results, Katz and Verma [8] gave a characterization for the}
\]
\[ \text{Cohen-Macaulayness of } T \text{ in terms of minimal reductions of } I. \]

In [20] (see Theorem 5.1), Viet generalized the result of Katz and Verma [op.cit.] to the case of equimultiple ideals in generalized Cohen-Macaulay rings.

This section proved that Theorem 5.1 [20] is also true for arbitrary Noetherian local rings by the following theorem.

**Theorem 4.1.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \( d > 0 \) with maximal ideal \( \mathfrak{m} \) and infinite residue field \( k = A/\mathfrak{m} \). Let \( I \) be an equimultiple ideal with \( \text{ht } I = h > 0 \). Suppose that \( J \) is a minimal reduction of \( I \) and \( r_J(I) \) is the reduction number of \( J \) with respect to \( J \). Let \( J' = (y_1, y_2, \ldots, y_d - h) \) be an ideal of \( A \) such that \( J = (J, J') \) is an \( \mathfrak{m} \)-primary ideal. \( T = A[It, t^{-1}] \) is the extended Rees algebra of \( I \). Then \( T \) is Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay and the following conditions are satisfied:

(i) \((JJ^n + J') \cap (I^{n+2} + J') = JJ^{n+1} + J' \) for all \( 0 \leq n \leq r_J(I) - 1 \).

(ii) \( J' \cap I^n + JJ^{n-1} + I^{n+1} = J'I^n + JJ^{n-1} + I^{n+1} \) for all \( 0 \leq n \leq r_J(I) \).

**Proof.** Denote by \( N \) the maximal homogeneous ideals of \( T \). Note that \( T \) is Cohen-Macaulay if and only if \( T_N \) is Cohen-Macaulay [7]. It is easily seen that \( B = (J, J')T_N \) is a parameter ideal of \( T_N \). Set \( I^{-1} = 0 \). Assume that \( r = r_J(I) \) is the reduction number of \( I \) with respect to \( J \). Then by the proof of Theorem 5.1 [20], we have

\[ l(T_N/B) = l(A/J) + \sum_{n=0}^{r-1} l((JJ^n + J') \cap (I^{n+2} + J')/(JJ^{n+1} + J')) \]

\[ + \sum_{n=0}^{r} l(J' \cap I^n + JJ^{n-1} + I^{n+1}/JJ^{n+1} + I^{n+1}). \]

It is easily seen that \( B \) is a reduction of \((I + J', It, t^{-1})\), \( e(I + J', It, t^{-1}) = e(B) \).

By Theorem 3.1, we have \( e(I + J', It, t^{-1}) = e(I + J') \). Since \( J \) is a reduction of \((I + J')\), \( e(J) = e(I + J') \). Thus, \( e(B) = e(J) \). Note that \( I \) is a parameter ideal of \( A \), \( l(A/J) \geq e(J) \). Hence \( l(T_N/B) = e(B) \) if and only if \( l(A/J) = e(J) \) and

\[ \sum_{n=0}^{r-1} l((JJ^n + J') \cap (I^{n+2} + J')/(JJ^{n+1} + J')) = \sum_{n=0}^{r} l(J' \cap I^n + JJ^{n-1} + I^{n+1}/JJ^{n+1} + I^{n+1}) = 0. \]

Thus, \( T_N \) and hence \( T \) are Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay and \((JJ^n + J') \cap (I^{n+2} + J') = JJ^{n+1} + J' \) for all \( 0 \leq n \leq r_J(I) - 1 \), and \( J' \cap I^n + JJ^{n-1} + I^{n+1} = J'I^n + JJ^{n-1} + I^{n+1} \) for all \( 0 \leq n \leq r_J(I) \). \( \square \)
Note 2: We emphasize that by [4, 4.5.7(a)], if \( G(I) \) is Cohen-Macaulay then \( A \) is Cohen-Macaulay. Hence the proof of Theorem 4.1 is immediate from [4, 4.5.7(a)] and [20, Theorem 5.1].

Now we will consider some corollaries of this theorem.

Back to the case where \( I \) is \( m \)-primary, then \( J' = 0 \) and we immediately get a generalization of Katz-Verma’s theorem [op.cit.] and Corollary 5.2 in [20] as follows:

**Corollary 4.2.** Let \((A, m)\) be a Noetherian local ring of dimension \( d > 0 \) with maximal ideal \( m \) and infinite residue field \( k = A/\mathfrak{m} \). Let \( I \) be an \( m \)-primary ideal. Suppose that \( J \) is a minimal reduction of \( I \) and \( r_J(I) \) is the reduction number of \( I \) with respect to \( J \). Then \( T = A[It, t^{-1}] \) is Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay and \( JI^n \cap I^{n+2} = JI^{n+1} \) for all \( 0 \leq n \leq r_J(I) - 1 \).

In the case of \( r_J(I) \leq 1 \), \( I^2 = JJ I \). Hence \( J \cap I^2 = I^2 = JJ I \). The condition of Corollary 4.2 is satisfied. Consequently, we have:

**Corollary 4.3.** Let \( I \) be \( m \)-primary ideal with \( r(I) \leq 1 \). Then \( G(I) \) is Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay.

This result is a development of the results in [15] and [11]. Note that \( r(I) \leq 1 \) means \( I^n = 2I^{n-1} \) for all \( n \geq 2 \). Therefore, \( J \cap I^n = 2I^{n-1} \) for all \( n \geq 1 \). Consequently, by Valabrega-Valla in [21], the Cohen-Macaulayness of \( A \) and \( G(I) \) are equivalent. Hence from [21] we also give Corollary 4.3.

Moreover, we have the following result in the case where \( I \) is equimultiple and \( r(I) \leq 1 \).

**Corollary 4.4.** Let \( I \) be an equimultiple ideal with \( ht I = h > 0 \). Suppose that \( J \) is a minimal reduction of \( I \) with \( r_J(I) \leq 1 \). \( J' = (y_1, y_2, \ldots, y_d - h) \) be an ideal of \( A \) such that \( J = (J, J') \) is an \( m \)-primary ideal. Then \( G(I) \) is Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay and \( J' \cap I = J'I \bmod J \).

**Proof.** For \( r_J(I) \leq 1 \), the condition (i) of Theorem 4.1 is equivalent to

\[
(J + J') \cap (I^2 + J') = JJ + J'.
\]  

(1)

Since \( r_J(I) \leq 1 \), \( I^2 = JJ I \). Hence \( (J + J') \cap (I^2 + J') = (I^2 + J') = JJ + J' \). So (1) holds. Now, the condition (ii) of Theorem 4.1 is

\[
J' \cap I^n + JJ^{n-1} + I^{n+1} = J'I^n + JJ^{n-1} + I^{n+1}
\]  

(2)

for \( n = 0, 1 \). It can be verified that \( n = 0 \), (2) is true. For \( n = 1 \), then (2) becomes

\[
J' \cap I + J + I^2 = J'I + J + I^2.
\]  

(3)

Since \( I^2 = JJ \subseteq J \), (3) is equivalent to \( J' \cap I = J'I \bmod J \). Hence by Theorem 4.1 we get the proof of this corollary. \( \square \)
Return to the case of \( I = \mathfrak{m} \) and \( r(\mathfrak{m}) \leq 2 \), we get a sharpening of the result in [12] as follows:

**Corollary 4.5.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \( d > 0 \) with maximal ideal \( \mathfrak{m} \) and \( r(\mathfrak{m}) \leq 2 \). Then \( G(\mathfrak{m}) \) is Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay.

**Proof.** Since \( r(\mathfrak{m}) \leq 2 \), there exists a minimal reduction \( J \) of \( \mathfrak{m} \) such that \( \mathfrak{m}^3 = J\mathfrak{m}^2 \). We need to check the conditions of Corollary 4.2 that \( J \cap \mathfrak{m}^2 = J\mathfrak{m} \) and \( J\mathfrak{m} \cap \mathfrak{m}^3 = J\mathfrak{m}^2 \). But first equation holds by Lemma 3 in [9]. Since \( \mathfrak{m}^3 = J\mathfrak{m}^2 \), \( J\mathfrak{m} \cap \mathfrak{m}^3 = \mathfrak{m}^3 = J\mathfrak{m}^2 \). Hence by Corollary 4.2 we get the proof. \( \Box \)

5 The extended Rees algebras with minimal multiplicity

Let \((A, \mathfrak{m})\) be a Cohen-Macaulay local ring of Krull dimension \( d > 0 \). Denote by \( \mu(I) \) is the minimum number of generators for an ideal \( I \). Then \( v(A) = \mu(\mathfrak{m}) = v \) is called the embedding dimension of \( A \). Abhyankar proved in [1] that \( v-d+1 \leq e(A) \). In the case of \( v-d+1 = e(A) \), then \( A \) is said to have minimal multiplicity. Sally in [11] showed that \( A \) has minimal multiplicity if and only if any minimal reduction \( J \) of \( \mathfrak{m} \), \( r_J(\mathfrak{m}) \leq 1 \). Katz and Verma [8] gave a criterion for the Cohen-Macaulayness with minimal multiplicity of extended Rees algebras of parameter ideals.

As an application of Section 3 and Section 4, in this section we will give some criteria for the Cohen-Macaulayness with minimal multiplicity of extended Rees algebras and associated graded rings of parameter ideals in local rings.

The following theorem is a version of Katz-Verma’s theorem [8] in local rings.

**Theorem 5.1.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \( d > 0 \) with maximal ideal \( \mathfrak{m} \) and infinite residue field \( k = A/\mathfrak{m} \). Let \( q \) be a parameter ideal of \( A \). Then \( T = A[qt, t^{-1}] \) is Cohen-Macaulay with minimal multiplicity if and only if either (i) \( A \) is Cohen-Macaulay with minimal multiplicity and \( \mathfrak{m}^2 = q\mathfrak{m} \) or (ii) \( A \) is regular and \( e(q + \mathfrak{m}^2; A) = 2 \).

**Proof.** If \( T \) is Cohen-Macaulay, \( A \) is Cohen-Macaulay by [4]. Hence by Theorem 5.2 in [8] we get the proof of Theorem 5.1. \( \Box \)

**Theorem 5.2.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of dimension \( d > 0 \) with maximal ideal \( \mathfrak{m} \) and infinite residue field \( k = A/\mathfrak{m} \). Let \( q \) be a parameter ideal of \( A \). Then \( G(q) \) is Cohen-Macaulay with minimal multiplicity if and only if \( A \) is Cohen-Macaulay and \( \mathfrak{m}^2 \subseteq q \).

**Proof.** Direct computation shows that \( v(G(q)) - \dim G(q) + 1 = l(A/\mathfrak{m}^2 + q) \). By Corollary 4.3, \( G(q) \) is Cohen-Macaulay if and only if \( A \) is Cohen-Macaulay.
Hence $G(q)$ is Cohen-Macaulay with minimal multiplicity if and only if $A$ is Cohen-Macaulay and $e(G(q)) = v(G(q)) - \dim G(q) + 1 = l(A/m^2 + q)$. Since $e(G(q)) = e(q)$ and $A$ is Cohen-Macaulay, $e(q) = l(A/q) \geq l(A/m^2 + q)$. Hence $G(q)$ is Cohen-Macaulay with minimal multiplicity if and only if $A$ is Cohen-Macaulay and $l(A/q) = l(A/m^2 + q)$. Since $q \subseteq m^2 + q$, $l(A/q) = l(A/m^2 + q)$ if and only if $q = m^2 + q$. This is equivalent to $m^2 \subseteq q$. So $G(q)$ is Cohen-Macaulay with minimal multiplicity if and only if $A$ is Cohen-Macaulay and $m^2 \subseteq q$.

As an immediate consequence of Theorem 5.2 and Lemma 5.1 [8] we have the following corollary.

**Corollary 5.3.** Let $q$ be a parameter ideal of $A$. When if $G(q)$ is Cohen-Macaulay with minimal multiplicity, then either (i) $A$ is Cohen-Macaulay with minimal multiplicity and $m^2 = qm$ or (ii) $A$ is a regular ring and $e(q; A) = 2$.

**Remark 5.4.** Since $G(q)$ is Cohen-Macaulay with minimal multiplicity, $m^2 \subseteq q$ by Theorem 5.2. So $e(m^2 + q) = e(q)$. Hence by Corollary 5.3 and Theorem 5.1, it follows that if $G(q)$ is Cohen-Macaulay with minimal multiplicity, then $A[qt, t^{-1}]$ is also Cohen-Macaulay with minimal multiplicity.

**References**


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