The Least Group Congruences on Eventually Regular Semigroups

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Abstract

In this paper, we give a group congruence and the least group congruence on eventually regular semigroup by using weakly self-conjugate subsemigroups which are analogous to the characterization of a group congruence on eventually regular semigroups considered by Rao and Lakshmi [4].

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1 Introduction

A semigroup $S$ is said to be eventually regular if every element of $S$ has a power which is regular by Edward [3]. Throughout this paper $S$ is an eventually regular semigroup and $E(S)$ denote the set of all idempotents of $S$. For every $a \in S$, $V(a) := \{ x \in S \mid a = axa, x = xax \}$ is the set of all inverses of element $a$ and $W(a) := \{ x \in S \mid x = xax \}$ is the set of all weak inverses of element $a$. For $a \in S$, by "$a^n$ is $a$-regular" we mean that $n$ is the smallest positive integer for which $a^n$ is regular. A semigroup $S$ is called $E$-semigroup if $E(S)$ forms a semigroup of $S$. A congruence $\rho$ on a semigroup $S$ is called a group congruence if $S/\rho$ is a group.

In 1983, Edward [3] characterized many results of eventually semigroups and finite semigroups. Basic properties and some results of eventually regular semigroups were given by Edward [3], Howie [1], Hall [5], Luo and Li [6] and Yang [7]. Rao and Lakshmi [4] described some group congruences on an eventually regular semigroup in which they use self-conjugate, that is
if for all \( a \in S, (a^n)' \in V(a^n) \) where \( a^n \) is a-regular, \( aHa^{-1}(a^n)' \subseteq H \) and \( a^{n-1}(a^n)'Ha \subseteq H \). In this paper, we investigated a group congruence and least group congruence on eventually regular semigroup and we replace the set \( V(a^n) \) as in [16] by the set of all weak inverses \( W(a^n) \) and replace self-conjugate subset \( H \) of an eventually regular semigroup \( S \) by weakly self-conjugate subset of its.

A subset \( H \) of a semigroup \( S \) is full if \( E(S) \subseteq H \). A subsemigroup \( H \) of an eventually regular semigroup \( S \) is called weakly self-conjugate if for all \( a \in S, (a^n)' \in W(a^n) \) where \( a^n \) is a-regular, \( aHa^{-1}(a^n)' \subseteq H \) and \( a^{n-1}(a^n)'Ha \subseteq H \). For any subset \( H \) of a semigroup \( S \), let \( H_\omega = \{ a \in S \mid ha \in H \text{ for some } h \in H \} \) which is called the closure of \( H \). If \( H \) is a subsemigroup of \( S \), then \( H \subseteq H_\omega \). A subsemigroup \( H \) of a semigroup \( S \) is closed if \( H = H_\omega \).

For any congruence \( \rho \) on a semigroup \( S \), the kernel of \( \rho \) is the set
\[
\ker \rho := \{ a \in S \mid a \rho \in E(S/\rho) \} = \{ a \in S \mid (a, a^2) \in \rho \}.
\]
If \( \rho \) is a group congruence on a semigroup \( S \), then \( a \in \ker \rho \) if and only if \( (a, e) \in \rho \) for some (all) \( e \in E(S) \). For basic concepts in semigroup theory, see [1].

## 2 Preliminary

The following results are used in this research.

**Lemma 2.1.** Let \( S \) be a semigroup and \( a \in S \). If \( a^n \) is a-regular then \( W(a^n) \neq \emptyset \).

**Proof.** It is easy to verify. \( \square \)

**Proposition 2.2.** If \( S \) is an \( E \)-semigroup, \( a^n \) is a-regular and \( e, f \in E(S) \), \( (a^n)' \in W(a^n) \), then
\[
\begin{align*}
(1) & \ e(a^n)', (a^n)'f, f(a^n)'e \in W(a^n), \\
(2) & \ a^n e(a^n)', (a^n)'ea^n \in E(S),
\end{align*}
\]
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(3) \(ae(a^{n-1})(a^n)'a^{n-1}(a^n)'ea \in E(S)\),
(4) \(a^{n-1}(a^n)', (a^n)'a^{n-1} \in W(a)\),
(5) \(fa^{n-1}(a^n)'e, fa^{n-1}(a^n)'e \in W(a)\),
(6) \(f(a^n)'a^{n-1}, (a^n)'a^{n-1}e, f(a^n)'a^{n-1}e \in W(a)\).

**Proposition 2.3.** If \(S\) is a semigroup, \(((ac)^n)' \in W((ac)^n)\) and \((ac)^n\) is ac-regular, then \(c(ac)^{n-1}((ac)^n)'a\) and \(c((ac)^n)'(ac)^{n-1}a \in E(S)\).

**Proof.** Let \(((ac)^n)' \in W((ac)^n)\). Then
\[
(c(ac)^{n-1}((ac)^n)'a)(c(ac)^{n-1}((ac)^n)'a) = c(ac)^{n-1}((ac)^n)'ac(ac)^{n-1}((ac)^n)'a = c(ac)^{n-1}[(ac)^n]'(ac)^n((ac)^n)'a = c(ac)^{n-1}((ac)^n)'a.
\]

Similarly, we have \(c((ac)^n)'(ac)^{n-1}a \in E(S)\). \(\Box\)

**Proposition 2.4.** If \(S\) is an E-semigroup, \(a^n\) is a-regular and \(b^m\) is b-regular, then \(W(b^m)W(a^n) \subseteq W(a^nb^m)\). If \(S\) is commutative then \(W(a^nb^m) = W((ab)^n)\).

**Proof.** Let \((a^n)' \in W(a^n), (b^m)' \in W(b^m)\). Then
\[
(b^m)'(a^n)'a^nb^m(b^m)'(a^n)' = (b^m)'b^m(b^m)'(a^n)'a^nb^m(b^m)'(a^n)'a^n(a^n)' = (b^m)'b^m(b^m)'(a^n)'a^n(a^n)' = (b^m)'(a^n)'.
\]

Therefore \((b^m)'(a^n)' \in W(a^nb^m)\) and so \(W(b^m)W(a^n) \subseteq W(a^nb^m)\). If \(S\) is commutative semigroup then \(W(a^nb^m) = W((ab)^n)\). \(\Box\)

## 3 Main Results

The next result, we show that \(\rho_H\) is a group congruence on an eventually regular semigroup which alternating as in \([4]\).

**Theorem 3.1.** [4] If \(S\) is an eventually regular semigroup then every group congruence on \(S\) is of the form \((a, b) \in \beta_H \iff ab^{m-1}(b^m)' \in H\) where \(H\) is full closed self-conjugate subsemigroup of \(S\) and \((b^m)' \in V(b^m)\).

The following theorem we give a group congruence on an eventually regular semigroup \(S\) by replace for some \((b^m)' \in V(b^m)\) by for all \((b^m)' \in W(b^m)\) as follows.
Theorem 3.2. If $S$ is an eventually regular semigroup and $H \in \mathcal{C}$, then a relation

$$\rho_H := \{(a,b) \in S \times S \mid ab^{m-1}(b^m)' \in H \text{ for all } (b^m)' \in W(b^m)$$

where $b^m$ is $b-$regular\}

is a group congruence on $S$.

Proof. Since $aa^{n-1}(a^n)' \in E(S) \subseteq H$ for all $a \in S$, $a^n$ is $a$-regular and $(a^n)' \in W(a^n)$, we have $a \rho_H a$.

To show that $\rho_H$ is symmetric, let $(a,b) \in \rho_H$, this implies that $ab^{m-1}(b^m)' \in H$, where $b^m$ is $b$-regular and $(b^m)' \in W(b^m)$. Let $(a^n)' \in W(a^n)$ where $(a^n)$ is $a$-regular. Since $H$ is weakly self-conjugate, $(ab^{n-1}(b^m)')(ba^{n-1}(a^n)') \in H$. Since $H = H_\omega$, we have $ba^{n-1}(a^n)' \in H$, so $\rho_H$ is symmetric.

If $ab^{m-1}(b^m)'$ and $bc^{k-1}(c^k)' \in H$ where $b^m$ is $b$-regular and $c^k$ is $c$-regular and $(b^m)' \in W(b^m), (c^k)' \in W(c^k)$, then $ab^{m-1}(b^m)'bc^{k-1}(c^k)' \in H$. Let $(a^n)' \in W(a^n)$ where $a^n$ be $a$-regular. As $H$ is a weakly self-conjugate, $a^n-1(a^n)'bc^{k-1}(c^k)'a \in H$, it follows that $c^{k-1}(c^k)'a \in H$. Again $c^{k-1}(c^k)'a \in H$ and $H = H_\omega$ imply $ac^{k-1}(c^k)' \in H$ which proves transitive of $\rho_H$. Hence $\rho_H$ is an equivalence relation.

To show that $\rho_H$ is a compatible, let $(a,b) \in \rho_H$ and $c \in S$. Then $ab^{m-1}(b^m)' \in H$ for all $b^m \in W(b^m)$ where $b^m$ is $b$-regular.

If $(a^n)' \in W(a^n)$ where $a^n$ is $a$-regular, we have $ba^{n-1}(a^n)' \in H$. Let $(bc)^k$ be $bc$-regular and $((bc)^k)' \in W((bc)^k)$ and $c^k$ be $c$-regular and $(c^k)' \in W(c^k), b^k$ be $b$-regular and $(b^k)' \in W(b^k)$. By Proposition 2.3, we have $c((bc)^{k-1}((bc)^k)'b \in E(S) \subseteq H$. By $H$ is a weakly self-conjugate, we have $ac((bc)^{k-1}((bc)^k)'ba^{n-1}(a^n)' \in H$ and so $ac((bc)^{k-1}((bc)^k)' \in H$, we get $\rho_H$ is a right compatible. We can show that $\rho_H$ is a left compatible. Hence $\rho_H$ is a congruence.

Fix $x \in H$. Let $a \in S$. By $H$ is a weakly self-conjugate, we have $axa^{n-1}(a^n)'$, $xa^{n-1}(a^n)' \in H$ where $a^n$ is $a$-regular and $(a^n)' \in W(a^n)$, so $(ax,a), (xa,a) \in \rho_H$. Hence $x \rho_H$ is the identity element of $S/\rho_H$. For any $a \in S$ and let $e \in E(S)$, if $a^n$ is $a$-regular, we have $a^n-1(a^n)' \rho_H a \rho_H = e \rho_H = a \rho_H(a^n-1)(a^n)' \rho_H$.

Hence $\rho_H$ is a group congruence on $S$. \hfill \Box

Next, we can prove a group congruence on an eventually regular semigroup $S$ by using full, weakly self-conjugate subsemigroup $H$ of $S$.

Theorem 3.3. If $S$ is an eventually regular semigroup and $H \in \mathcal{C}$, then a relation

$$\rho_H^* := \{(a,b) \in S \times S \mid xa = by \text{ for some } x, y \in H\}$$

is a group congruence on $S$. 
Proof. To show that $\rho^*_H$ is a congruence on $S$, let $a, b, c \in S$. Let $(a^n)' \in W(a^n)$ where $a^n$ is $a$-regular. Since $H$ is full, $(a^n)(a^n)', a(a^{n-1})(a^n)' \in E(S) \subseteq H$. Note that $[a^n(a^n)']a = a[(a^{n-1})(a^n)']$, we have $a\rho^*_H a$.

Suppose that $a\rho^*_H b$, then there exist $x, y \in H$ such that $xa = by$. Let $(a^n)' \in W(a^n)$ where $a^n$ is $a$-regular and $(b^m)' \in W(b^m)$ where $b^m$ is $b$-regular. Then $[(a^n)(a^n)']byb^{m-1}(b^m)'b = a\rho^*_H b$. Since $x\rho^*_H y$, there exist $(a^n)(a^n)'xa)b^{m-1}(b^m)'b \in H$, we have $b\rho^*_H a$.

To show that $\rho^*_H$ is transitive, let $a\rho^*_H b$ and $b\rho^*_H c$. Then there exist $x, y, z, w \in H$ such that $xa = by$ and $zb = cw$. Thus $(zx)a = c(wy)$ and $zx, wy \in H$, it follows that $a\rho^*_H c$ and $\rho^*_H$ is transitive.

Next, we want to show that $\rho^*_H$ is a compatible, let $a\rho^*_H b$ and $c \in S$. Then there exist $x, y \in H$ such that $xa = by$. Let $(b^n)' \in W(b^n)$ and $(c^n)' \in W(c^n)$ where $b^n$ is $b$-regular and $c^n$ is $c$-regular. Then $b\rho^*_H c$. Clearly, by (1), $(c^n)\rho^*_H (b^n)'x = b (c^n)'(b^n)'(b^m)'byc$. Since $H$ is a weakly self-conjugate, we have $bc(b^n)'(b^n)'x, c^{-1}(c^n)'(b^n)'byc \in H$. Hence $\rho^*_H$ is a right compatible. Similarly, we can show that $\rho^*_H$ is a left compatible, so $\rho^*_H$ is a congruence on $S$.

Finally, we shall show that $S/\rho^*_H$ is a group. Fix $x \in H$. Claim that $x\rho^*_H$ is the identity element of $S/\rho^*_H$. Let $a \in S$ and $(a^n)' \in W(a^n)$ where $a^n$ is $a$-regular. Since $xa\rho^*_H a \in H$, we have $[xa\rho^*_H a]a = (ax)(a^n)'a$, so $(a, ax) \in \rho^*_H$. Since $xa\rho^*_H a \in H$, $x(a\rho^*_H a) = (xa)(a^n)'a$ and so $(a, xa) \in \rho^*_H$. Hence $x\rho^*_H$ is the identity of $S/\rho^*_H$.

Clearly, $x\rho^*_H y \rho^*_H = e\rho^*_H$ for all $x, y \in H, e \in E(S)$. Then $(a^n-1(a^n)')\rho_H = (a^n-1(a^n))\rho_H = e\rho^*_H = (a^n-1(a^n)')\rho^*_H = a\rho^*_H (a^n-1(a^n)')\rho^*_H$.

Therefore $a^n-1(a^n)\rho^*_H$ is an inverse of $a\rho^*_H$. Hence $S/\rho^*_H$ is a group.

Remark. From Theorem 3.3, we see that $H \subseteq \text{Ker}\rho^*_H$ for every $H \in \mathcal{C}$.

**Lemma 3.4.** Let $S$ be an eventually regular semigroup.

1. If $H \in \mathcal{C}$ then $\text{Ker}\rho^*_H = H_\omega$.
2. If $H \in \overline{\mathcal{C}}$ then $\text{Ker}\rho^*_H = H_\omega$.
3. If $\rho$ is a group congruence on $S$ then $\text{Ker}\rho \in \mathcal{C} \subseteq \overline{\mathcal{C}}$ and $\rho = \rho_{\text{Ker}\rho}$.

**Proof.** (1) Suppose that $H \in \mathcal{C}$. By Theorem 3.3, $\rho^*_H$ is a group congruence on $S$. Let $a \in \text{Ker}\rho^*_H$. Then $(a, e) \in \rho^*_H$ for all $e \in E(S)$. Let $e \in E(S)$. Then $xa = ey$ for some $x, y \in H$. Since $ey \in H$, we get $xa \in H$. Thus $a \in H_\omega$.

Conversely, let $a \in H_\omega$. Then there exists $h \in H$ such that $ha \in H$. For any $(a^n)' \in W(a^n)$ where $a^n$ is $a$-regular $[(a^n(a^n)')h]a = ((a^n')a)h$. Since $(a^n)(a^n)'h, (a^n)'H \in H$, so $((a^n)(a^n)'a) \in \rho^*_H$ and $a \in \text{Ker}\rho^*_H$. Therefore $\text{Ker}\rho^*_H = H_\omega$.

(2) Clearly, by (1), $H = H_\omega = \text{Ker}\rho^*_H$.

(3) Let $e \in E(S)$. Then $(e, e) \in \rho$, so $e \in \text{Ker}\rho$. Thus $\text{Ker}\rho$ is full. Let $x \in \text{Ker}\rho$. Then $(x, e) \in \rho$ for all $e \in E(S)$. Let $a \in S, (a^n)' \in W(a^n)$ where
$a^n$ is $a$-regular. Then $(a^{n-1}(a^n)'xa, a^{n-1}(a^n)'ea) \in \rho$, and $a^{n-1}(a^n)'xap = (a^{n-1}(a^n)'eap = a^{n-1}(a^n)'epepa = a^{n-1}(a^n)'aep = a^{n-1}(a^n)'aep$ where $e\rho$ is the identity element in $S/\rho$. Then $(a^{n-1}(a^n)'xa, a^{n-1}(a^n)'a) \in \rho$, so $a^{n-1}(a^n)'xa \in \text{Ker}\rho$. Similarly, we can show that $ax(a^{n-1})(a^n)' \in \text{Ker}\rho$.

Hence $\text{Ker}\rho$ is a weakly self-conjugate subset of $S$.

Next, we shall show that $\text{Ker}\rho$ is a subsemigroup of $S$. Let $a, b \in \text{Ker}\rho$. Then $ap = ep, bp = ep$ for all $e \in E(S)$. Thus $(ab)\rho = a\rho b = epep = ep$ for all $e \in E(S)$. Hence $ab \in \text{Ker}\rho$. That is, $\text{Ker}\rho \in \mathcal{C}$.

Let $x \in (\text{Ker}\rho)_\omega$. Then there exists $y \in \text{Ker}\rho$ such that $yx \in \text{Ker}\rho$. Thus $(yx)\rho = ep$ for all $e \in E(S)$ and $ypxp = (yx)\rho = ep$. Since $y \in \text{Ker}\rho$, we have $yp = ep$. Hence $xp = ep$, so $x \in \text{Ker}\rho$. Therefore $(\text{Ker}\rho)_\omega = \text{Ker}\rho$, so $\text{Ker}\rho \in \mathcal{C}$.

Finally, we shall show that $\rho = \rho_{\text{Ker}\rho}^*$. Let $(a, b) \in \rho$ and $(a^n)' \in W(a^n)$ where $a^n$ is $a$-regular. Then $(a(a^n)'a^{n-1}, b(a^n)'a^{n-1}) \in \rho$. We get $b(a^n)'a^{n-1} \in \text{Ker}\rho$, and $(b(a^n)'a^{n-1}) = b((a^n)'a^{n-1}) = (a^n)'a^{n-1}$). Then $(a, b) \in \rho_{\text{Ker}\rho}^*$ and so $\rho \subseteq \rho_{\text{Ker}\rho}^*$.

Suppose that $(a, b) \in \rho_{\text{Ker}\rho}^*$. Then $xa = by$ for some $x, y \in \text{Ker}\rho$. Thus $xp = ep = yp$ for all $e \in E(S)$. Since $\rho$ is a group congruence, $b(y)p = b(e)p = b\rho$ and so $(a, b) \in \rho$. Hence $\rho = \rho_{\text{Ker}\rho}^*$.

**Corollary 3.5.** Let $S$ be an eventually regular semigroup. Then $\rho$ is a group congruence on $S$ if and only if there exists $K \in \overline{\mathcal{C}}$ such that $\rho = \rho_K^*$ where $K = \text{Ker}\rho$.

**Proof.** It is similar to the proof of Theorem 3.3 and Lemma 3.4(3). \hfill \Box

**Lemma 3.6.** Let $S$ be an eventually regular semigroup.

1. If $H \subseteq K \subseteq S$ then $\rho_H^* \subseteq \rho_K^*$.
2. If $H, K \in \mathcal{C}$ such that $\rho_H^* \subseteq \rho_K^*$, then $H \subseteq K$.

(hence for $H, K \in \overline{\mathcal{C}}, H \subseteq K$ if and only if $\rho_H^* \subseteq \rho_K^*$).

**Proof.** The proof as in [2]. \hfill \Box

By Lemma 3.4 and 3.6, we have the least group congruence on an eventually regular semigroups.

**Theorem 3.7.** Let $S$ be an eventually regular semigroup. If $U$ is the smallest element in $\mathcal{C}$ then $\rho_U^*$ is the least group congruence on $S$.

**Proof.** The proof as in [2]. \hfill \Box

Combine the Theorem 3.2, 3.3, Corollary 3.5 and Lemma 3.6, we obtain
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Theorem 3.8. Let $S$ be an eventually regular semigroup and let $\rho$ be a group congruence on $S$ with $H := \text{Ker}\rho$. Assume that $a^n$ is a-regular and $b^m$ is $b$-regular. Then the following statements are equivalent.

1. $a \rho b$.
2. For all $(a^n)' \in W(a^n), ba^{n-1}(a^n)' \in H$.
3. For all $(a^n)' \in W(a^n), a^{n-1}(a^n)'b \in H$.
4. For all $(b^m)' \in W(b^m), b^{m-1}(b^m)'a \in H$.
5. For all $(b^m)' \in W(b^m)$ there exists $x \in H$, $axb^{m-1}(b^m)' \in H$.
6. For all $(a^n)' \in W(a^n)$ there exists $x \in H$, $bxa^{n-1}(a^n)' \in H$.
7. For all $(a^n)' \in W(a^n)$ there exists $x \in H$, $a^{-1}(a^n)'xb \in H$.
8. For all $(b^m)' \in W(b^m)$ there exists $x \in H$, $b^{m-1}(b^m)'xa \in H$.
9. There exist $x, y \in H$ such that $xa = by$.
10. There exist $x, y \in H$ such that $ax = yb$.
11. $HaH \cap HbH \neq \emptyset$.

Proof. Let $a, b \in S$ and $(a^n)' \in W(a^n)'$ where $a^n$ be $a$-regular and $(b^m)' \in W(b^m)$ where $b^m$ be $b$-regular.

(1) $\Rightarrow$ (2) By Theorem 3.2, $\rho$ is symmetric and so $ba^{n-1}(a^n)' \in H$.

(2) $\Rightarrow$ (3) Suppose that $ba^{n-1}(a^n)' \in H$. Since $H$ is a weakly self-conjugate, we have $b^{m-1}(b^m)'ba^{n-1}(a^n)'b \in H$. By Lemma 3.4(3), we have $H = \text{Ker}\rho \in \mathcal{C}$. Since $H$ is full and $H = H_\omega$, we have $b^{m-1}(b^m)'b \in E(S) \subseteq H$, so $a^{-1}(a^n)'b \in H$.

(3) $\Rightarrow$ (4) Since $(b^m)(b^m)' \in E(S) \subseteq H$ for all $(b^m)' \in W(b^m)$ where $b^m$ is $b$-inversive, we have $a^{-1}(a^n)'b(b^{m-1})(b^m)'a = a^{-1}(a^n)'b^{m-1}(b^m)'a \in H$. Now $a^{-1}(a^n)'b \in H$ implies $b^{m-1}(b^m)'a \in H$. If $b^{m-1}(b^m)'a \in H$, we have $b(b^{m-1}(b^m)'a b^{m-1}(b^m))' \in H$, so $ab^{m-1}(b^m)' \in H$ which proves (1), (2), (3), (4) are equivalent.

(5) $\Rightarrow$ (6) Suppose that $axb^{m-1}(b^m)' \in H$ for some $x \in H$. Since $x \in H$, we have $xb^{m-1}(b^m)'bx \in H$ and so $axb^{m-1}(b^m)'bxa^{-1}(a^n)' \in H$. Since $H = H_\omega$, we have $bxa^{-1}(a^n)' \in H$.

(6) $\Rightarrow$ (7) If $bxa^{-1}(a^n)' \in H$ for some $x \in H$ then $bxa^{-1}(a^n)'xb \in H$. Since $H$ is a weakly self-conjugate, $b^{m-1}(b^m)'bx a^{-1}(a^n)'xb \in H$. Since $H = H_\omega$, we have $a^{-1}(a^n)'xb \in H$.

(7) $\Rightarrow$ (8) Suppose that $a^{-1}(a^n)'xb \in H$ for some $x \in H$. Since $b^{m-1}(b^m)'xa(a^{-1})(a^n)'xb \in H$ and $H = H_\omega$, we have $b^{m-1}(b^m)'xa \in H$.

(8) $\Rightarrow$ (9) Suppose that $b^{m-1}(b^m)'xa \in H$ for some $x \in H$. Then $b^{m-1}(b^m)'xa = y$ where $y \in H$ and $b^{m}(b^m)'xa = by$. Put $b^{m}(b^m)'x = x_1$. Then $x_1a = by$ for some $x_1, y \in H$.

(9) $\Rightarrow$ (10) Suppose that $xa = by$ for some $x, y \in H$ implies

$$a^n(a^n)'xab^{m-1}(b^m)'b = a^n(a^n)'byb^{m-1}(b^m)'b \quad \text{and}$$

$$a[a^{-1}(a^n)'xab^{m-1}(b^m)'b] = [a^n(a^n)'byb^{m-1}(b^m)]b.$$
y_1b for some \( x_1, y_1 \in H \).

(10) \Rightarrow (11) Suppose that \( ax = yb \) for some \( x, y \in H \). Then \( xaxy = xyby \in HaH \cap HbH \), so \( HaH \cap HbH \neq \emptyset \).

(11) \Rightarrow (5) Suppose that \( HaH \cap HbH \neq \emptyset \) implies \( h_1ah_2 = t_1bt_2 \) for some \( h_1, h_2, t_1, t_2 \in H \). Now \( h_1ah_2 = t_1bt_2 \) implies

\[
\begin{align*}
  a^n(a^n)'h_1ah_2b^{m-1}(b^m)'b &= a^n(a^n)'t_1bt_2b^{m-1}(b^m)'b \\
  a[a^{n-1}(a^n)'h_1ah_2b^{m-1}(b^m)']b &= [a^n(a^n)'t_1bt_2b^{m-1}(b^m)']b.
\end{align*}
\]

Hence \( ax = yb \) for some \( x, y \in H \), which implies \( axb^{m-1}(b^m)' = yb^{m}(b^m)' \in H \). Hence (5) and (11) are equivalent.

(1) \Rightarrow (9) If \( ab^{m-1}(b^m)' = y \in H \) then \( ab^{m-1}(b^m)'b = yb \), so \( ax = yb \) for some \( x, y \in H \).

(5) \Rightarrow (4) Now, \( axb^{m-1}(b^m)' \in H \) implies \( a^{n-1}(a^n)'axb^{m-1}(b^m)'a \in H \). So \( b^{m-1}(b^m)'a \in H \), which completes the proof. \( \square \)

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