Fuzzy Dot $BCK/BCI$-Algebras

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Abstract

In this paper the notion of fuzzy dot $BCK$-subalgebra is introduced. We state and prove some theorem in fuzzy dot $BCK$-subalgebra and level subalgebras.

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1 Introduction and preliminaries

Processing of certain information especially inferences based on certain information, therefore is based on classical two-valued logic. Logic appears in a ‘scared’ form (resp., a ‘profane’) which is dominant in proof theory (resp., model theory). The role of logic in mathematics and computer science is two fold as a tool for applications in both areas, and a technique for laying the foundations.

Non-classical logic including many-valued logic, fuzzy logic, etc., takes the advantage of the classical logic to handle information with various facets of uncertainty, such as fuzziness, randomness, and so on. Non-classical logic has become a formal and useful tool for computer science to deal with fuzzy information and uncertain information. Among all kinds of uncertainties, incomparability is an important one which can be encountered in our life.

In recent years, motivated by both theory and application, the study of $t$-norm-based logic systems and the corresponding pseudo-logic systems has become a greater focus in the field of logic. Here, $t$-norm-based algebraic investigations were first to the corresponding algebraic investigations, and in the case of pseudo-logic systems, algebraic development was first to the corresponding logical development. As it is well known, $BCK/BCI$-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki
BCI-algebras are generalizations of BCK-algebras. Most of the algebras related to the $t$-norm based logic, such as MTL-algebras, BL-algebras [3, 4], hoop, MV-algebras and Boolean algebras et al., are extensions of BCK-algebras.

The concept of fuzzy sets was first initiated by Zadeh [11]. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc.

In the present paper, we introduced the concept of fuzzy dot BCK-subalgebras and fuzzy dot topological BCK-algebras and study this structure. We state and prove some theorem discussed in fuzzy dot BCK-subalgebras and level subalgebras and give the relationship between this notion and fuzzy BCK-subalgebras. Finally some of Fosters results on homomorphic images and inverse images in fuzzy dot topological BCK-algebras are studied.

If $\mu$ is a fuzzy set in a $BCK/BCI$-algebra $X$. Then $\mu$ is called a fuzzy $BCK/BCI$-subalgebra (algebra) of $X$ if

$$\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$ [9].

**Definition 1.1.** [8] A fuzzy topology on a set $X$ is a family $\tau$ of fuzzy sets in $X$ which satisfies the following condition:

(i) For $c \in [0, 1], k_c \in \tau$, where $k_c$ has a constant membership function,

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$,

(iii) $\tau$ closed under arbitrary union, which means that if $A_j \in \tau$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \tau$.

The pair $(X, \tau)$ is called a fuzzy topological space and members of $\tau$ are called open fuzzy sets.

**Definition 1.2.** [8] Let $A$ be a fuzzy set in $X$ and $\tau$ a fuzzy topology on $X$. Then the induced fuzzy topology on $A$ is the family of fuzzy subsets of $A$ which are the intersection with $A$ of $\tau$-open fuzzy sets in $X$. The induced fuzzy topology is denoted by $\tau_A$, and the pair $(A, \tau_A)$ is called a fuzzy subspace of $(X, \tau)$.

**Definition 1.3.** [8] Let $(X, \tau)$ and $(Y, \upsilon)$ be two fuzzy topological space. A mapping $f$ of $(X, \tau)$ into $(Y, \upsilon)$ is fuzzy continuous if for each open fuzzy set $U$ in $\upsilon$ the inverse image $f^{-1}(U)$ is in $\tau$. Conversely, $f$ is fuzzy open if for each fuzzy set $V$ in $\tau$, the image $f(V)$ is in $\upsilon$.

Let $(A, \tau_A)$ and $(B, \upsilon_B)$ be fuzzy subspace of fuzzy topological spaces $(X, \tau)$ and $(Y, \upsilon)$ respectively, and let $f$ be a mapping from $(X, \tau)$ to $(Y, \upsilon)$.
Then $f$ is a mapping of $(A, \tau_A)$ into $(B, \upsilon_B)$ if $f(A) \subseteq B$. Furthermore $f$ is relatively fuzzy continuous if for each open fuzzy set $V'$ in $\upsilon_B$ the intersection $f^{-1}(V') \cap A$ is in $\tau_A$. Conversely, $f$ is relatively fuzzy open if for each open fuzzy set $U'$, the image $f(U')$ is in $\upsilon_B$.

**Lemma 1.4.** [2] Let $(A, \tau_A), (B, \upsilon_B)$ be fuzzy subspace of fuzzy topological space $(X, \tau), (Y, \upsilon)$ respectively, and let $f$ be a fuzzy continuous mapping of $(X, \tau)$ into $(Y, \upsilon)$ such that $f(A) \subset B$. Then $f$ is a relatively fuzzy continuous mapping of $(A, \tau_A)$ into $(B, \upsilon_B)$.

### 2 Main Results

From now on $X$ is a $BCK$-algebra, unless otherwise is stated.

**Definition 2.1.** Let $\mu$ be a fuzzy set in $X$. Then $\mu$ is called a fuzzy dot $BCK$-subalgebra (algebra) of $X$ if

$$\mu(x \ast y) \geq \mu(x) \cdot \mu(y)$$

for all $x, y \in X$.

**Example 2.2.** Let $X = \{0, a, b, c\}$ be a set with the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>b</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X, \ast, 0)$ is a $BCI$-algebra.

Define a fuzzy set $\mu : X \to [0, 1]$ by $\mu(0) = 0.5$, $\mu(x) = 0.7$ for all $x \in \{a, b, c\}$. Then $\mu$ is a fuzzy dot $BCI$-subalgebra of $X$.

Note that every fuzzy $BCK$-subalgebra of $X$ is a fuzzy dot $BCK$-subalgebra of $X$, but the converse is not true.

In fact, the fuzzy dot $BCI$-subalgebra in above example is not a fuzzy $BCI$-subalgebra, since

$$\mu(a \ast a) = \mu(0) = 0.5 < 0.7 = \mu(a) = \min\{\mu(a), \mu(a)\}.$$
Proof. For all \( x \in X \), we have \( x \ast x = 0 \). Hence \( \mu(0) = \mu(x \ast x) \geq \mu(x) \cdot \mu(x) = (\mu(x))^2 \).

**Theorem 2.4.** Let \( A \) be a fuzzy dot \( BCK \)-subalgebra of \( X \). If there exists a sequence \( \{x_n\} \) in \( X \), such that

\[
\lim_{n \to \infty} (\mu(x_n))^2 = 1
\]

Then \( \mu(0) = 1 \).

Proof. By Lemma 2.3, we have \( \mu(0) \geq (\mu(x))^2 \), for all \( x \in X \), thus \( \mu(0) \geq (\mu(x_n))^2 \), for every positive integer \( n \). Consider

\[
1 \geq \mu(0) \geq \lim_{n \to \infty} (\mu(x_n))^2 = 1.
\]

Hence \( \mu(0) = 1 \).

**Theorem 2.5.** Let \( \mu \) and \( \nu \) are fuzzy dot \( BCK \)-subalgebras of \( X \). Then \( \mu \cap \nu \) is a fuzzy dot \( BCK \)-subalgebras of \( X \).

Proof. Let \( x, y \in \mu \cap \nu \). Then \( x, y \in \mu \) and \( \nu \), since \( \mu \) and \( \nu \) are fuzzy dot \( BCK \)-subalgebras of \( X \) by above theorem we have:

\[
(\mu \cap \nu)(x \ast y) = \min\{\mu(x \ast y), \nu(x \ast y)\}
\geq \min\{\mu(x), \mu(y), \nu(x), \nu(y)\}
\geq (\min\{\mu(x), \nu(x)\}) \cdot (\min\{\mu(y), \nu(y)\})
= ((\mu \cap \nu)(x)) \cdot ((\mu \cap \nu)(y))
\]

Which proves the theorem.

**Corollary 2.6.** Let \( \{\mu_i\mid i \in \Lambda\} \) be a family of fuzzy dot \( BCK \)-subalgebras of \( X \). Then \( \bigcap_{i \in \Lambda} \mu_i \) is also a fuzzy dot \( BCK \)-subalgebras of \( X \).

**Definition 2.7.** Let \( \mu \) be a fuzzy set in \( X \) and \( \lambda \in [0, 1] \). Then the level \( BCK \)-subalgebra \( U(\mu; \lambda) \) of \( \mu \) and strong level \( BCK \)-subalgebra \( U(\mu; >, \lambda) \) of \( \mu \) are defined as following:

\[
U(\mu; \lambda) := \{x \in X \mid \mu(x) \geq \lambda\},
\]

\[
U(\mu; >, \lambda) := \{x \in X \mid \mu(x) > \lambda\}.
\]
Remark. If $\mu$ is a fuzzy dot subalgebra of $X$, then $U(\mu; \lambda)$ or $U(\mu; >, \lambda)$ need not be a subalgebra of $X$. Since, in Example 2.2, define fuzzy subset $\mu$ by: $\mu(0) = 0.5$, $\mu(a) = 0.6$, $\mu(b) = 0.4$ and $\mu(c) = 0.3$, then $\mu$ is a fuzzy dot subalgebra of $X$. But $U(\mu; >, 0.35) = \{0, a, b\} = U(\mu; 0.4)$ is not a subalgebra of $X$, since $a, b \in U(\mu; >, 0.35)$ and $a \ast b = c \notin U(\mu; >, 0.35)$.

Theorem 2.8. Let $\mu$ be a fuzzy dot $BCK$-subalgebra of $X$. Then $U(\mu; 1) = \{x \in X \mid \mu(x) = 1\}$ is either empty or is a subalgebra of $X$.

Proof. If $x, y \in U(\mu; 1)$, then $\mu(x \ast y) \geq \mu(x), \mu(y) = 1$. Hence $\mu(x \ast y) = 1$, which implies that $x \ast y \in U(\mu; 1)$. Consequently, $U(\mu; 1)$ is a subalgebra of $X$.

Proposition 2.9. Let $f$ be a $BCK$-homomorphism from $X$ into $Y$ and $\mu$ be a fuzzy dot $BCK$-subalgebra of $Y$. Then the inverse image $f^{-1}(\mu)$ of $\mu$ is a fuzzy dot $BCK$-subalgebra of $X$.

Proof. Let $x, y \in X$. Then

$$f^{-1}(\mu)(x \ast y) = \mu(f(x \ast y))$$

$$= \mu(f(x) \ast f(y))$$

$$\geq \mu(f(x)), \mu(f(y))$$

$$= f^{-1}(\mu)(x), f^{-1}(\mu)(y).$$

Then $f^{-1}(\mu)$ is a fuzzy dot $BCK$-subalgebra of $X$.

Proposition 2.10. Let $f$ be a $BCK$-homomorphism from $X$ onto $Y$ and $\mu$ be a fuzzy dot $BCK$-subalgebra of $X$ with the sup property. Then the image $f(\mu)$ of $\mu$ is a fuzzy dot $BCK$-subalgebra of $Y$.

Proof. Let $x, y \in Y$, $A = f^{-1}(x)$, $B = f^{-1}(y)$ and $C = f^{-1}(x \ast y)$. Consider

$$A \ast B = \{t = a \ast b \mid a \in A, b \in B\}.$$

It is clear that $C \subseteq A \ast B$. We have

$$f(\mu)(x \ast y) = \sup_{t \in f^{-1}(x \ast y)} \mu(t)$$

$$= \sup_{t \in C} \mu(t)$$

$$\geq \sup_{t \in A \ast B} \mu(t).$$
\[ \sup_{x \in A, y \in B} \mu(x) \geq \sup_{x \in A, y \in B} \mu(x) \cdot \mu(y). \]

We know that the operation \( : [0, 1] \times [0, 1] \to [0, 1] \) is continuous, then for any \( \epsilon > 0 \) there exists a \( \delta > 0 \), such that if \( \bar{x} \geq \sup_{x \in A} \mu(x) - \delta \) and \( \bar{y} \geq \sup_{y \in B} \mu(y) - \delta \).

Then \( \bar{x} \cdot \bar{y} \geq \sup_{x \in A} \mu(x) \cdot \sup_{y \in B} \mu(y) - \epsilon \). Choose \( a \in A \) and \( b \in B \) such that

\[
\mu(a) \geq \sup_{x \in A} \mu(x) - \delta \quad \text{and} \quad \mu(b) \geq \sup_{y \in B} \mu(y) - \delta.
\]

Thus \( \mu(a) \cdot \mu(b) \geq \sup_{x \in A} \mu(x) \cdot \sup_{y \in B} \mu(y) - \epsilon \). Therefore

\[
f(\mu)(x \cdot y) \geq \sup_{x \in A, y \in B} \mu(x) \cdot \mu(y) \geq \sup_{x \in A} \mu(x) \cdot \sup_{y \in B} \mu(y) = f(\mu)(x) \cdot f(\mu)(y).
\]

Hence \( f(\mu) \) is a fuzzy dot \( BCK \)-subalgebra of \( Y \).

**Definition 2.11.** Let \( \rho \) be a fuzzy subset of \( X \). The strongest fuzzy \( \rho \)-relation on \( X \) is a fuzzy subset \( \mu_\rho \) of \( X \times X \) given by \( \mu_\rho(x, y) = \rho(x) \cdot \rho(y) \), for all \( x, y \in X \).

**Theorem 2.12.** Let \( \mu_\rho \) be the strongest fuzzy \( \rho \)-relation on \( X \), where \( \rho \) is a fuzzy subset of \( X \). If \( \rho \) is a fuzzy dot \( BCK \)-subalgebra of \( X \), then \( \mu_\rho \) is a fuzzy dot \( BCK \)-subalgebra of \( X \times X \).

**Proof.** Let \( \rho \) be a fuzzy dot \( BCK \)-subalgebra of \( X \), \( x_1, x_2, y_1, y_2 \in X \). We have

\[
\mu_\rho((x_1, y_1) \cdot (x_2, y_2)) = \mu_\rho(x_1 \cdot y_1, x_2 \cdot y_2) = \rho(x_1 \cdot x_2) \cdot \rho(y_1 \cdot y_2) \geq (\rho(x_1) \cdot \rho(x_2)) \cdot (\rho(y_1) \cdot \rho(y_2)) = \mu_\rho(x_1, y_1) \cdot \mu_\rho(x_2, y_2).
\]

Therefore \( \mu_\rho \) is a fuzzy dot \( BCK \)-subalgebra of \( X \times X \).
**Definition 2.13.** Let $\rho$ be a fuzzy subset of $X$. A fuzzy relation $\mu$ on $X$ is called a fuzzy $\rho$-product relation if $\mu(x * y) \geq \rho(x) \rho(y)$, for all $x, y \in X$.

**Definition 2.14.** Let $\rho$ be a fuzzy subset of $X$. A fuzzy relation $\mu$ on $X$ is called a left fuzzy relation on $\rho$ if $\mu(x * y) \geq \rho(x)$, for all $x, y \in X$.

Similarly we can define a right fuzzy relation on $\rho$. It is clear that a left (right) fuzzy relation on $\rho$ is a fuzzy $\rho$-product relation.

**Theorem 2.15.** Let $\mu$ be a left fuzzy relation on fuzzy relation $\rho$ of $X$. If $\mu$ is a fuzzy dot $BC^K$-subalgebra of $X \times X$, then $\rho$ is a fuzzy dot $BC^K$-subalgebra of $X$.

**Proof.** Let $\mu$ left fuzzy relation $\mu$ on $\rho$ is a fuzzy dot $BC^K$-subalgebra of $X$. Thus

$$
\rho(x_1 * x_2) = \mu(x_1 * x_2, y_1 * y_2) = \mu((x_1, y_1) * (x_2, y_2)) \geq \mu(x_1, y_1) \mu(x_2, y_2) = \rho(x_1) \rho(x_2).
$$

for all $x_1, x_2, y_1, y_2 \in X$. Therefore $\rho$ is a fuzzy dot subalgebra of $X$.

**Theorem 2.16.** Let $\mu$ be a fuzzy relation on $X$ satisfying the inequality $\mu(x, y) \leq \mu(x, 0)$, for all $x, y \in X$. Define $\rho_t$ by $\rho_t(x) = \mu(x, t)$, for all $x \in X$ and $t \in X$. If $\mu$ is a fuzzy dot $BC^K$-subalgebra of $X \times X$, then $\rho_t$ is a fuzzy dot $BC^K$-subalgebra of $X$, for all $t \in X$.

**Proof.** Let $x, y, t \in X$. Then

$$
\rho_t(x * y) = \mu(x * y, t) = \mu(x * y, t * 0) = \mu((x, t) * (y, 0)) \geq \mu(x, t) \mu(y, 0) \geq \mu(x, t) \mu(y, t) = \rho_t(x) \rho_t(y).
$$

Therefore $\rho_t$ is a fuzzy dot $BC^K$-subalgebra of $X$.

**Theorem 2.17.** Let $\mu$ be a fuzzy relation on $X$ and $\rho_\mu$ be a fuzzy subset of $X$ given by $\rho_\mu(x) = \inf_{y \in X} \mu(x, y), \mu(y, x)$, for all $x \in X$. If $\mu$ is a fuzzy dot
$BCK$-subalgebra of $X \times X$ satisfying the equality $\mu(x, 0) = 1 = \mu(0, x)$, for all $x \in X$. Then $\rho_\mu$ is a fuzzy dot $BCK$-subalgebra of $X$, for all $t \in X$.

**Proof.** Let $x, y, z \in X$. Then

$$
\mu(x * y, z) = \mu(x * y, z * 0) = \mu((x, z) * (y, 0)) \\
\geq \mu(x, z) \mu(y, 0) = \mu(x, z).
$$

And

$$
\mu(z, x * y) = \mu(z * 0, x * y) = \mu((z, x) * (0, y)) \\
\geq \mu(z, x) \mu(0, y) = \mu(z, x).
$$

Therefore

$$
\mu(x * y, z) \mu(z, x * y) \geq \mu(x, z) \mu(z, x) \\
\geq (\mu(x, z) \mu(z, x)) (\mu(y, z) \mu(z, y)).
$$

Now, consider

$$
\rho_\mu(x * y) = \inf_{z \in X} \mu(x * y, z) \mu(z, x * y) \\
\geq (\inf_{z \in X} (\mu(x, z) \mu(z, x))) (\inf_{z \in X} (\mu(y, z) \mu(z, y))) \\
= \rho_\mu(x) \rho_\mu(y).
$$

Therefore $\rho_\mu$ is a fuzzy dot $BCK$-subalgebra of $X$.

**Definition 2.18.** A fuzzy map $f$ from a set $X$ to a set $Y$ is an ordinary map from $X$ to the set of all fuzzy subsets of $Y$ satisfying the following conditions:

(i) for all $x \in X$, there exists $y_x \in X$ such that $(f(x))(y_x) = 1$,

(ii) for all $x \in X$, $f(x)(y_1) = f(x)(y_2)$ implies $y_1 = y_2$.

We can see that a fuzzy map $f$ from $X$ to $Y$

(i) gives rise to a unique ordinary map $\mu_f : X \times X \to I$, given by $\mu_f(x * y) = f(x)(y)$.

(ii) gives a unique ordinary map $f_1 : X \to Y$ defined as $f_1(x) = y_x$.

Now we can generalize the notion of homomorphism to fuzzy homomorphism.
**Definition 2.19.** Let $X, Y$ be $BCK/BCI$-algebras. A fuzzy map $f$ from a set $X$ to a set $Y$ is called a fuzzy dot homomorphism if
\[ \mu_f(x_1 \ast x_2, y) = \sup_{y=y_1 \ast y_2} \mu_f(x_1, y_1) \mu_f(x_2, y_2), \]
for all $x_1, x_2 \in X$ and $y \in Y$.

**Proposition 2.20.** Let $f : X \to Y$ be a fuzzy homomorphism of $BCK/BCI$-algebras. Then
(i) $\mu_f(x_1 \ast x_2, y_1 \ast y_2) \geq \mu_f(x_1, y_1) \mu_f(x_2, y_2)$, for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.
(ii) $\mu_f(0, 0) = 1$.
(iii) $\mu_f(0 \ast x, 0 \ast y) \geq \mu_f(x, y)$, for all $x \in X$ and $y \in Y$.

**Proof.** (i) For any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have
\[
\mu_f(x_1 \ast x_2, y_1 \ast y_2) = \sup_{y_1 \ast y_2 = y_1' \ast y_2'} \mu_f(x_1, y_1') \mu_f(x_2, y_2') 
\geq \mu_f(x_1, y_1) \mu_f(x_2, y_2).
\]

(ii) Let $s \in X$. Since $f$ is a fuzzy homomorphism then there exists a $y_s \in Y$ such that $\mu_f(x, y_s) = 1$, then we get that
\[
\mu_f(0, 0) = \mu_f(s \ast s, y_s \ast y_s) \geq \mu_f(x, y_s) \mu_f(x, y_s) = 1.
\]

(iii) By (i) and (ii) is clear.

For any $BCK/BCI$-algebra $X$ and any element $a \in X$ we denote by $R_a$ the right translation of $X$ defined by $R_a(x) = x \ast a$ for all $x \in X$. It is clear that $R_0(x) = 0 = R_x(x)$ for all $x \in X$.

**Definition 2.21.** Let $\tau$ be a fuzzy topology on $X$ and $D$ be a fuzzy dot $BCK$-subalgebra of $X$ with induced topology $\tau_D$. Then $D$ is called a fuzzy dot topological $BCK/BCI$-algebra of $X$ if for each $a \in X$ the mapping $R_a : (D, \tau_D) \to (D, \tau_D)$ is relatively fuzzy continuous.

**Theorem 2.22.** Let $X$ and $Y$ be two $BCK/BCI$-algebras, $f : X \to Y$ be a $BCK/BCI$-homomorphism. Let $\tau$ and $\nu$ be the fuzzy dot topologies on $X$ and $Y$ respectively, such that $\tau = f^{-1}(\nu)$. Let $G$ be a fuzzy dot topological $BCK$-subalgebra of $Y$ with membership function $\mu_G$. Then $f^{-1}(G)$ is a fuzzy dot topological $BCK/BCI$-algebra of $X$ with membership function $\mu_{f^{-1}(G)}$.

**Proof.** We must show that, for each $a \in X$, the mapping
is relatively fuzzy continuous. Let $U$ be any open fuzzy set in $\tau_{f^{-1}(G)}$ on $f^{-1}(G)$. Since $f$ is a fuzzy continuous mapping from $(X, \tau)$ into $(Y, \upsilon)$, from Lemma 1.4 follows that $f$ is a relatively fuzzy continuous mapping of $(f^{-1}(G), \tau_{f^{-1}(G)})$ into $(G, \upsilon_G)$. Note that there exists an open fuzzy set $V \in \upsilon_G$ such that $f^{-1}(V) = U$. The membership function of $R^{-1}_{a}(U)$ is given by

$$
\mu_{R^{-1}_{a}(U)}(x) = \mu_U(R_a(x)) = \mu_U(x * a) = \mu_{f^{-1}(V)}(x * a) = \mu_V(f(x * a)) = \mu_V(f(x) * f(a)).
$$

Since $G$ is a fuzzy dot topological $BCK/BCI$-algebra of $Y$, the mapping

$$
R_b: (G, \upsilon_G) \rightarrow (G, \upsilon_G)
$$

is relatively fuzzy continuous for each $b \in Y$. Hence

$$
\mu_{R^{-1}_{a}(U)}(x) = \mu_V(f(x) * f(a)) = \mu_V(R_{f(a)}(f(x))) = \mu_{R^{-1}_{f(a)}(V)}(f(x)) = \mu_{R^{-1}_{f(a)}(V)}(f(x)).
$$

which implies that $R^{-1}_{a}(U) = f^{-1}(R^{-1}_{f(a)}(V))$ therefore

$$
R^{-1}_{a}(U) \cap f^{-1}(G) = f^{-1}(R^{-1}_{f(a)}(V)) \cap f^{-1}(G)
$$

is a open in the relative fuzzy dot topology on $f^{-1}(G)$.

**Theorem 2.23.** Given $BCK/BCI$-algebras $X$ and $Y$ and a $BCK/BCI$-homomorphism $f$ from $X$ onto $Y$, let $\tau$ be the fuzzy dot topology on $X$ and $\upsilon$ be the fuzzy dot topology on $Y$ such that $f(\tau) = \upsilon$. Let $D$ be a fuzzy dot topological $BCK/BCI$-algebra of $X$. If the membership function $\mu_D$ of $D$ is a $f$-invariant, then $f(D)$ is a fuzzy dot topological $BCK/BCI$-algebra of $Y$.

**Proof.** It is enough to show that the mapping

$$
R_b: (f(D), \upsilon_{f(D)}) \rightarrow (f(D), \upsilon_{f(D)})
$$

is relatively fuzzy continuous, for all $b \in Y$. It is clear that $f$ is a relatively fuzzy open mapping, since for $U \in \tau_D$ there exists $U' \in \tau$ such that $U = U' \cap D$, by $f$-invariance of $\mu_D$ we have

$$
f(U) = f(U) \cap f(D) \in \upsilon_{f(D)}.
$$
Let $V'$ be an open fuzzy set in $v_{f(D)}$. For any $b \in Y$ by hypothesis there exists $a \in X$ such that $b = f(a)$. Thus

$$\mu_{f^{-1}(R_b^{-1}(V'))}(x) = \mu_{f^{-1}(R_{f(a)}^{-1}(V'))}(x) = \mu_{R_{f(a)}^{-1}(V')}(f(x))$$

$$= \mu_{V'}(R_{f(a)}(f(x))) = \mu_{V'}(f(x) \ast (f(a)))$$

$$= \mu_{V'}(f(x \ast a)) = \mu_{f^{-1}(V')}(x \ast a)$$

$$= \mu_{f^{-1}(V')(R_a(x))} = \mu_{R_a^{-1}(f^{-1}(V'))}(x)$$

which implies that $f^{-1}(R_b^{-1}(V')) = R_a^{-1}(f^{-1}(V'))$. By hypothesis, $R_a$ is a relatively fuzzy continuous mapping from $(D, \tau_D)$ to $(D, \tau_D)$ and $f$ is a relatively fuzzy continuous mapping from $(D, \tau_D)$ to $(f(D), v_{f(D)})$. Therefore

$$f^{-1}(R_b^{-1}(V')) \cap G = R_a^{-1}(f^{-1}(V')) \cap D$$

is open in $\tau_D$. Since $f$ is relatively fuzzy open, then

$$f(f^{-1}(R_b^{-1}(V')) \cap D) = R_b^{-1}(V') \cap f(D)$$

is open in $v_{f(D)}$.

## 3 Conclusion

In the present paper, we have introduced the concept of fuzzy dot subalgebras of $BCK/BCI$-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other fuzzy algebraic systems such as groups, semigroups, rings, nearrings, semirings (hemirings), lattices and Lie algebras.

It is our hope that this work would other foundations for further study of the theory of $BCK/BCI$-algebras. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis.

In our future study of fuzzy structure of $BCK/BCI$-algebras, may be the following topics should be considered:

1. To establish a fuzzy dot ideals of $BCK/BCI$-algebras;
2. To consider the structure of quotient $BCK/BCI$-algebras by using these fuzzy dot ideals;
3. To get more results in fuzzy dot $BCK/BCI$-algebras and application.

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