A Kind of Graph Structure on Von-Neumann Regular Rings

Ai-Hua Li and Qi-Sheng Li

1,2Department of Mathematics, Jishou University Jishou, Hunan 416000, P.R. China
1 lah0928@jsu.edu.cn
2 nlqlqsh@jsu.edu.cn

Abstract

A kind of graph structure of a ring $R$ can be defined as the undirected graph $\Gamma_N(R)$ that two nonzero elements $x$ and $y$ of $R$ are adjacent if and only if $xy$ is a nil-element. If $R$ is a von Neumann regular ring or a commutative ring, then $\Gamma_N(R)$ is connected, the diameter of $\Gamma_N(R)$ is at most 3, and the girth of $\Gamma_N(R)$ is no more than 4. Moreover, if $R$ is nonreduced, then the girth of $\Gamma_N(R)$ is 3 or \( \infty \). For a finite commutative ring $R$, we show that the edge chromatic number of $\Gamma_N(R)$ is equal to the maximum degree of $\Gamma_N(R)$, unless $R$ is a nilpotent ring with even order. We also prove that, with two exceptions, if $R$ is a finite reduced commutative ring and $S$ is a commutative ring which is not an integral domain and $\Gamma_N(R) \simeq \Gamma_N(S)$, then $R \simeq S$. If $R$ and $S$ are finite nonreduced commutative rings and $\Gamma_N(R) \simeq \Gamma_N(S)$, then $|R| = |S|$ and $|N(R)| = |N(S)|$.

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1. Introduction

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The concept of zero-divisor graph of a commutative ring was introduced by Beck in [1] to discuss the colorings of rings. In [2], Anderson and Livingston introduced and studied the zero-divisor graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors and two vertices $x$ and $y$ are adjacent if and only if $xy = 0$. Recently, Redmond in [3] has extended this concept to any arbitrary ring. Since the zero-divisor graph can help us to study the algebraic properties of rings using graph theoretical tools, it has been studied extensively in recent years (e.g., see [4-10]). In [11], Chen defined a kind of graph structure of rings. He let all elements of ring $R$ be the vertices of the graph and two vertices $x$ and $y$ are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all nil-elements of $R$. He discussed the vertex coloring of the graph and proved that if all the nil-elements of $R$ are in the center of $R$, then the vertex chromatic number and the clique number of the graph are equal. Motivated by the work of Chen, for a ring $R$, we define a kind of new undirected graph $\Gamma_N(R)$ with vertex set $Z_N(R)^*$, and satisfying that $x$ and $y$ in $Z_N(R)^*$ are adjacent if and only if $xy \in N(R)$ (or equivalently, $yx \in N(R)$), where $Z_N(R) = \{x \in R | xy \in N(R) \text{ for some } y \in R^* \}$, and $X^* = X \setminus \{0\}$ for any subset $X$ of $R$. Obviously, our definition is slightly different from the one defined by Chen and it is easy to see that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of the graph $\Gamma_N(R)$.

Recall that a ring $R$ is (von Neumann) regular provided that for every $x \in R$ there exists $y \in R$ such that $xyx = x$. For a ring $R$, let $Z(R)$ denote the set of all zero-divisors of $R$, and let $|X|$ denote the cardinality of the subset $X$ of $R$. Recall that a ring $R$ is nonreduced if $N(R) \neq 0$. Generally, $N(R)$ is not an ideal of $R$. Also recall that is called a null ring if $R^2 = \{0\}$. For a graph $G$, let $V(G)$ denote the set of vertices of a graph $G$. Recall that an undirected graph $G$ is connected if there exits a path between any two distinct vertices. For distinct vertices $x$ and $y$, let $d(x, y)$ be the length of the shortest path from $x$ to $y$ ($d(x, y) = \infty$ if there is no such path). The diameter of $G$ is $\text{diam}(G) = \sup \{d(x, y) | x \text{ and } y \text{ are distinct vertices of } G \}$. The girth of $G$, denoted by $\text{gr}(G)$, is defined as the length of the shortest cycle in $G$ ($\text{gr}(G) = \infty$ if $G$ contains no cycles). The degree $d(x)$ of a vertex $x$ in $G$ is the number of edges incident to $x$. We denote the minimum and maximum degree of vertices of $G$ by $\delta(G)$ and $\Delta(G)$, respectively. A $k$-edge coloring of a graph $G$ is an assignment of $k$ colors $\{1, \cdots, k\}$ to the edges of $G$ such that no two adjacent edges have the same color. The edge chromatic number $\chi'(G)$ of a graph $G$, is the minimum $k$ for which $G$ has a $k$-edge coloring. For other
In this paper, we study the interplay between the ring-theoretic properties of \( R \) and the graph-theoretic properties of \( \Gamma_N(R) \). Note that if \( R \) is reduced, then \( \Gamma_N(R) \) is actually the general zero-divisor graph \( \Gamma(R) \), so we focus our main attention on nonreduced rings. Section 2 is concerned with the basic property of \( \Gamma_N(R) \). For a commutative ring \( R \), we prove that \( \Gamma_N(R) \) is connected, the diameter of \( \Gamma_N(R) \) is at most 3, and the girth of \( \Gamma_N(R) \) is no more than 4. Moreover, if \( R \) is nonreduced, then the diameter of \( \Gamma_N(R) \) is no more than 2 and the girth of \( \Gamma_N(R) \) is 3 or \( \infty \). Similarly, we study the connectivity, the diameter and the girth of \( \Gamma_N(R) \) of regular rings, and determine all regular nonreduced rings for which \( \Gamma_N(R) \) is a star graph.

Section 3 deals with finite commutative rings. We first study the edge chromatic number of \( \Gamma_N(R) \). We show that if \( R \) is a finite commutative ring, then the edge chromatic number of \( \Gamma_N(R) \) is equal to the maximum degree of \( \Gamma_N(R) \), unless \( R \) is a nilpotent ring with even order. In particular, if \( R \) is a finite commutative ring with identity, then \( \chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R)) \).

When \( \Gamma_N(R) \cong \Gamma_N(S) \) implies \( R \cong S \) is a very interesting question. For the zero-divisor graph \( \Gamma(R) \) of finite commutative rings, this question has been studied extensively in [14] and [5]. In this section we prove that if \( R \) is a finite reduced commutative ring and \( S \) is a commutative ring which is not an integral domain, then \( \Gamma_N(R) \cong \Gamma_N(S) \) if and only if \( R \cong S \), unless \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( S \) is a null ring of order 3, or \( R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \) and \( S \) is isomorphic to either \( \mathbb{Z}_4 \) or \( \mathbb{Z}_2[x]/(x^2) \).

2. Basic Properties of \( \Gamma_N(R) \)

The main purpose of this section is to characterize the connectedness, diameter and girth of \( \Gamma_N(R) \) of commutative rings and regular rings. In addition, we give some descriptions when \( \Gamma_N(R) \) contains no cycles for those rings. We start with the following remarks.

Remark 1. Ring \( R \) is a domain if and only if \( \Gamma_N(R) \) is empty. In fact, if \( R \) is a domain, then \( \mathcal{Z}_N(R)^* = \emptyset \), and hence \( \Gamma_N(R) \) is empty. Conversely, if \( \Gamma_N(R) \) is empty, then \( R \) has no nonzero zero-divisors by the definition of \( \Gamma_N(R) \). So \( R \) is a domain.

Remark 2. Assume \( R \) be a regular ring with identity, then any element of
$R/Z(R)$ is unit. In fact, note that for any $x \in R$, there exists $y \in R$ such that $xyx = x$. Now if $x \notin Z(R)$, then $xy = 1$, hence $x$ is unit.

**Remark 3.** Let $R$ be a commutative ring or a regular ring. If $R$ is reduced, then $|R| \leq |Z(R)|^2 = |Z_N(R)|^2$, otherwise $|Z_N(R)^*| = |R| - 1$. In fact, if $R$ is reduced, then $\Gamma_N(R)$ is actually the zero-divisor graph $\Gamma(R)$. So we have $|R| \leq |Z(R)|^2 = |Z_N(R)|^2$ by [6, Remark 1]. If $R$ is a nonreduced commutative ring, then $r$ and $x$ are adjacent, for any $r \in R^*$ and $x \in N(R)^*$. Therefore $Z_N(R) = R$. Now we suppose that $R$ is a nonreduced regular ring. Let $r$ be an arbitrary element of $R^*$. If $r \in Z(R)$, then $Z_N(R) = R$ since $Z(R) \subseteq Z_N(R)$, and hence we are done. Otherwise, by Remark 2, $r$ is a unit and we denote the inverse element by $r^{-1}$. Let $x$ be a nonzero nil-element. It is easy to see that $r^{-1}x \neq 0, r$, and $r$ is adjacent to $r^{-1}x$. Thus $r \in Z_N(R)$, and hence $|Z_N(R)| = |R|.$

**Remark 4.** Note that if there exists a nonzero nil-element in the center of $R$, then all nonzero elements of $R$ are adjacent to it. Thus for any ring $R$, if the center of $R$ contains at least one nonzero nil-element, then (1) $\Gamma_N(R)$ is connected and $\text{diam}(\Gamma_N(R)) \leq 2$, (2) $\text{gr}(\Gamma_N(R)) = 3$, or $\infty$.

**Theorem 2.1.** Let $R$ be a commutative ring. Then the following hold:

1. $\Gamma_N(R)$ is connected.
2. $\text{diam}(\Gamma_N(R)) \leq 3$.
3. If $\Gamma_N(R)$ contains a cycle, then $\text{gr}(\Gamma_N(R)) \leq 4$. Moreover, if $R$ is nonreduced, then $\text{gr}(\Gamma_N(R)) = 3$.

**Proof.** We first prove (1) and (2). Assume that $x, y \in Z_N(R)^*$ and $x \neq y$. Then there exist $x_1, y_1 \in Z_N(R)^*$ such that $xx_1, yy_1 \in N(R)$ by the definition of $\Gamma_N(R)$.

Case 1. If $xy \in N(R)$, then $x$ and $y$ are adjacent and hence $d(x, y) = 1$.

Case 2. Assume that $xy \notin N(R)$. If $x_1y_1 \neq 0$, then $x - x_1y_1 - y$ is the shortest path from $x$ to $y$ and hence $d(x, y) = 2$. If $x_1y_1 = 0$, then $x - x_1 - y_1 - y$ is a path (not necessarily the shortest path) from $x$ to $y$ and hence $d(x, y) \leq 3$. This completes the proof of (1) and (2).

(3) If $R$ is reduced, then $\Gamma_N(R)$ is actually the zero-divisor graph $\Gamma(R)$. Thus if $\Gamma_N(R)$ contains a cycle, then by [4, (1.4)], $\text{gr}(\Gamma_N(R)) \leq 4$. Now we suppose that $R$ is nonreduced and $\Gamma_N(R)$ contains a cycle of length $n$, where $n \geq 4$. Without loss of generality, assume that $x_0 - x_1 - \cdots - x_{n-1} - x_0$ is such a cycle. Consider the following two cases:
Case 1. Assume that all \( x_i \)'s are not nilpotent elements. Since \( R \) is nonreduced, there exists a nonzero nilpotent element \( a \) such that \( a - x_{i-1} - x_i - a \) is a triangle, where \( 1 \leq i \leq n \). Thus \( \text{gr}(\Gamma_N(R)) = 3 \).

Case 2. Assume that at least one of \( x_i \)'s is nilpotent, say \( x_0 \). Note that \( x_0 \) is adjacent to each \( x_i \), where \( 1 \leq i \leq n-1 \), thus \( x_0 - x_i - x_{i+1} - x_0 \) is a triangle, where \( 1 \leq i \leq n-2 \), and hence \( \text{gr}(\Gamma_N(R)) = 3 \). ■

We give a explicit description when \( \Gamma_N(R) \) contains no cycles for a nonreduced commutative ring.

**Theorem 2.2.** Assume that \( R \) is a nonreduced commutative ring and \( \Gamma_N(R) \) is not a singleton. Then the following statements are equivalent:

1. \( \text{gr}(\Gamma_N(R)) = \infty \);
2. \( \Gamma_N(R) \) is a star graph;
3. \( R \) is either a null ring of order 3, or \( N(R) \) is a prime ideal of \( R \) with \( |N(R)| = 2 \).

**Proof.** (1) \( \Leftrightarrow \) (2). Clearly, if \( \Gamma_N(R) \) is a star graph then its diameter is \( \infty \). Conversely, if \( \text{gr}(\Gamma_N(R)) = \infty \), note that for a nonreduced commutative ring there exists a vertex which adjacent to all other vertices, then \( \Gamma_N(R) \) is a star graph.

(1) \( \Leftrightarrow \) (3). The sufficiency is clear. For the other direction, assume \( \text{gr}(\Gamma_N(R)) = \infty \). If \( |N(R)| \geq 3 \), then \( \Gamma_N(R) \) contains a triangle, a contradiction. Hence \( |N(R)^*| \leq 2 \). Consider the following two cases:

Case 1. If \( |N(R)^*| = 2 \), then it is clear that \( R = N(R) \). Assume \( R = \{0, a, b\} \). If \( a^2 \neq 0 \), then \( a^2, a, a + a^2 \) are pairwise distinct elements of \( R^* \), a contradiction (noting that if \( a + a^2 = 0 \), then \( a \) is not nilpotent). So \( a^2 = b^2 = 0 \). If \( ab \neq 0 \), then assume without loss that \( ab = a \). Since \( 0 = ab^2 = abb = ab = a \), a contradiction occurs. So \( ab = 0 \) and \( R \) is a null ring.

Case 2. If \( |N(R)^*| = 1 \). Given any \( x, y \in R \setminus N(R) \). If \( x = y \), then \( xy = x^2 \notin N(R) \), for otherwise \( x \in N(R) \), a contradiction. If \( x \neq y \), then \( xy \notin N(R) \), for otherwise, there is a triangle \( x - y - a - y \), where \( a \in N(R)^* \), a contradiction again. Hence \( N(R) \) is a prime ideal. ■

In the following, we will deal with the zero-divisor graphs of regular rings.

**Theorem 2.3.** Let \( R \) be a regular ring with identity 1. Then the following hold:

1. \( \Gamma_N(R) \) is connected.
(2) diam(\(\Gamma_N(R)\)) \leq 3.

(3) If \(\Gamma_N(R)\) contains a cycle, then \(\text{gr}(\Gamma_N(R)) \leq 4\). Moreover, if \(R\) is nonreduced, then \(\text{gr}(\Gamma_N(R)) = 3\).

**Proof.** We first prove (1) and (2). Note that if \(R\) is reduced, then by the definition of \(\Gamma_N(R)\), it becomes a special case of \(\Gamma(R)\) in [3, Theorem 2.3]. Now assume that \(R\) is nonreduced, and hence \(\mathbb{Z}_N(R) = R\) by Remark 3. Since \(R\) is regular, we have that any element of \(R\) is exactly either a unit or in \(Z(R)\) by Remark 2. Assume that \(x\) is a nonzero nilpotent element and \(n\) is the smallest positive integer such that \(x^n = 0\). For any two distinct vertices \(r_1\) and \(r_2\) of \(\Gamma_N(R)\), we consider the following cases:

Case 1: \(r_1, r_2 \in R \setminus Z(R)\). Note that \(x r_1^{-1} \neq 0, r_1,\) and \(r_2^{-1} x^{n-1} \neq 0, r_2\). It is easy to see that \(r_1 - x r_1^{-1} - r_2^{-1} x^{n-1} - r_2\) is a path from \(r_1\) to \(r_2\) and hence \(\text{diam}(\Gamma_N(R)) \leq 3\).

Case 2: \(r_1 \in R \setminus Z(R), r_2 \in Z(R)^*\). If \(r_2\) is a left zero-divisor, then there exists a nonzero element \(b\) such that \(r_2 b = 0\). If \(b x = 0\), then \(r_1 - x r_1^{-1} - b - r_2\) is a path from \(r_1\) to \(r_2\). If \(b x \neq 0\), then \(r_1 - x^{n-1} r_1^{-1} - b x - r_2\) is a path from \(r_1\) to \(r_2\). Similarly, we can prove it when \(r_2\) is a right zero-divisor.

Case 3: \(r_1, r_2 \in Z(R)^*\). If \(r_1 r_2 \in N(R)\), then \(r_1 - r_2\) is a path from \(r_1\) to \(r_2\). Now assume that \(r_1 r_2 \notin N(R)\). If there exist \(a, b \in R^*\) such that \(r_1 a = br_2 = 0\), then we obtain a path \(r_1 - a - b - r_2\) when \(ab = 0\) or a path \(r_1 - ab - r_2\) when \(ab \neq 0\). If there exist \(a, b \in R^*\) such that \(r_1 a = r_2 b = 0\) and \(br_2 \neq 0\) then there is a path \(r_1 - ab r_2 - br_2 - r_2\) when \(ab r_2 \neq 0\) or a path \(r_1 - a - br_2 - r_2\) when \(ab r_2 = 0\).

Thus \(\Gamma_N(R)\) is connected and \(\text{diam}(\Gamma_N(R)) \leq 3\).

(3). We first suppose that \(R\) is reduced and \(\Gamma_N(R)\) contains a cycle. Without loss of generality, suppose that \(x_1 - x_2 - x_3 - \cdots - x_n - x_1\) is a cycle with \(n > 4\) and \(x_1, \cdots, x_n\) are pairwise distinct. Let \(I = \text{ann}(x_1) \cap \text{ann}(x_3)\). Note that for any \(a, b \in R\), if \(ab = 0\), then \(ba = 0\) since \(R\) is reduced. Thus \(x_2 \in I\) and \(I\) is an ideal of \(R\). If \(I\) contains more than one nonzero elements, then let \(0 \neq y \in I \setminus \{x\}\) and we obtain a cycle \(x_1 - x_2 - x_3 - y - x_1\), and thus we are done. So we assume that each of ideals \(\text{ann}(x_1) \cap \text{ann}(x_3), \text{ann}(x_3) \cap \text{ann}(x_5)\) contains an unique nonzero element. Consider the value \(x_2 x_4\). If \(x_2 x_4 = 0\), then \(x_2 - x_3 - x_4 - x_2\) is a cycle. If \(x_2 x_4 \neq 0\), since \(x_2 x_4 \in \text{ann}(x_1) \cap \text{ann}(x_3)\) and \(x_2 x_4 \in \text{ann}(x_3) \cap \text{ann}(x_5)\), then \(x_2 = x_2 x_4 = x_4\), a contradiction. Thus if \(R\) is reduced and \(\Gamma_N(R)\) contains a cycle, then \(\text{gr}(\Gamma_N(R)) \leq 4\).

Now we assume that \(R\) is nonreduced and \(\Gamma_N(R)\) contains a cycle. Suppose
that $|N(R)^*| \geq 2$. Let $a, b$ be distinct elements in $N(R)^*$ and let $k, l$ be minimum positive integers such that $a^k = 0$ and $b^l = 0$ respectively. Thus we obtain a cycle $a - a^{k-1}b^{l-1} - b - 1 - a$ when $a^{k-1}b^{l-1} \neq 0$ or a cycle $a^{k-1} - b^{l-1} - a^{k-1}$ when $a^{k-1}b^{l-1} = 0$ and $a^{k-1} \neq b^{l-1}$, or a cycle $a - a^{k-1}(= b^{l-1}) - b - 1 - a$ when $a^{k-1} = b^{l-1}$. So we conclude that if $R$ contains more than one nonzero nilpotent element, then $\text{gr}(\Gamma_N(R)) = 3$. Now suppose that $R$ contains just one nonzero nilpotent element $x$. We choose an edge $c - d$ of the cycle satisfies that $x \neq c, d$. Obviously, we may assume $cd = 0$ or $cd = x$. If $cd = x$ and $xc \notin N(R)$, then easily to check that $dx, xc$ and $x$ are pairwise distinct, and hence $x - dx - xc - x$ is a triangle. If $cd = x$ and $xc \in N(R)$, note that $dx \neq c$ (otherwise, $c^2 = 0$, a contradiction), then we obtain a cycle $x - c - dx - x$ whence $dx \neq 0, x$, or a cycle $x - c - d - x$ otherwise. Now we consider the case of $cd = 0$. To describe conveniently, we denote $\text{ann}(x)$ the union of the left and right annihilator of $x$ for any nonzero element of $R$ in the rest proof.

Case 1: $x \in \text{ann}(c) \cap \text{ann}(d)$. Then $x - c - d - x$ is a triangle.

Case 2: $x \in \text{ann}(c)$, $x \notin \text{ann}(d)$. Then it is easy to check that $x - c - dx - x$ is a triangle whence $dx \neq x$, otherwise, $x - c - d - x$ is a triangle.

Case 3: $x \notin \text{ann}(c)$, $x \notin \text{ann}(d)$. If $xc = dx$, then $xc - d - c - dx$ is a triangle since $xc, c$, and $d$ are pairwise distinct. Assume $xc \neq dx$. If $xc = x$, then $x - c - dx - x$ is a triangle since $c$, $dx$, and $x$ are pairwise distinct. Similarly, we obtain a cycle $x - xc - d - x$ when $dx = x$. If $xc \neq x \neq dx$, then easily see that $x - xc - dx - x$ is a cycle.

Thus we conclude that if $R$ is nonreduced and contains a cycle then $\text{gr}(\Gamma_N(R)) = 3$.

In the next theorem, we determine all nonreduced regular rings $R$ for which $\Gamma_N(R)$ is a star graph.

**Theorem 2.4.** Assume that $R$ is a nonreduced regular ring, and $\Gamma_N(R)$ is a star graph. Then the following hold:

1. $\Gamma_N(R)$ has exactly two vertices if and only if $R$ is a nilpotent ring with order 3.

2. If $\Gamma_N(R)$ has at least three vertices then $N(R)$ is a prime ideal of $R$ with $|N(R)| = 2$.

**Proof.** (1) By Remark 3, $|R| = |V(\Gamma_N(R))| + 1$. Thus $|R| = 3$, which implies the additive group $R$ is cyclic and $R$ is commutative. So $R$ is a null ring of
order 3. The proof of the other direction is straightforward.

(2) We first claim that $R$ has only one nonzero nilpotent element and it is that vertex of $\Gamma_N(R)$ which is adjacent to all other vertices (we call it the special vertex of $\Gamma_N(R)$). To see this, we suppose that $a$ is a nonzero nilpotent element of $R$ and $n$ is the smallest positive integer such that $a^n = 0$. Since $\Gamma_N(R)$ is a star graph, we have $n \leq 3$. Suppose $n = 3$. Since $a$ and $a^2$ are adjacent, either $a$ or $a^2$ is the special vertex of $\Gamma_N(R)$. Let $b \in R - \{0, a, a^2\}$. If $a$ is the special vertex, then $ab \in N(R)$. If $ab = 0$, then $a^2b = 0$, which implies $b$ and $a^2$ are adjacent, a contradiction. If $ab \neq 0$, note that $ba^2 = 0$ and hence $b$ and $a^2$ are adjacent, a contradiction), then $b$ is a nonzero-divisor. Thus $R \neq Z(R)$, and hence $R$ has identity 1 by [6, Lemma 2]. This implies $a - a^2 - 1 - a$ is a triangle, we also get a contradiction. Now we suppose that $a^2$ is the special vertex. Then $a^2b \in N(R)$. If $a^2b = 0$, note that $ab \neq 0, a, a^2$, then $a - ab - a^2 - a$ is a triangle, a contradiction. If $a^2b \neq 0$, then $ba^2 \neq 0$. In fact, if $ba^2 = 0$, note that $ba \notin N(R)$ since $ab \notin N(R)$, then $a - ba - a^2 - a$ is a triangle. Thus $ba^2 \neq 0$, and hence $b$ is a nonzero-divisor. Therefore $R$ has identity 1 and $a - a^2 - 1 - a$ is a triangle, a contradiction.

So we conclude that the order of any nonzero nilpotent element of $R$ is 2. Furthermore, if $x$ is the special vertex of $\Gamma_N(R)$ and $x \neq a$, then it is easily checked that $x - a - (x + a) - x$ is a triangle, a contradiction. This implies every nilpotent element is the special vertex of $\Gamma_N(R)$, and hence $R$ has only one nonzero nilpotent element since $\Gamma_N(R)$ is a star graph.

Now we assume that $a$ is the unique nonzero nilpotent element of $R$ with $a^2 = 0$. Since $\Gamma_N(R)$ is a star graph and $a$ is the vertex which is adjacent to all other vertices, $ab, ba \in N(R)$ for any $b \in R$. Thus $\{0, a\}$ is an ideal of $R$. Also since $\Gamma_N(R)$ is a star graph, we see that $N(R)$ is a prime ideal. The proof of the other direction is straightforward.

3. Some results on finite commutative rings

In this section, all rings are finite commutative. We first investigate the edge chromatic number of $\Gamma_N(R)$ and prove that if $R$ is a finite commutative ring, then $\chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R))$ unless $R$ is a nilpotent ring with even order. Recall that the edge chromatic number $\chi'(G)$ of a graph $G$ is the minimum number of colors required to color the edges of $G$ in such a way that the edges incident to one vertex have different colors, so, $\chi'(G) \geq \Delta(G)$. It is well known that if $G$ is a simple graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$
see [12, Theorem 6.2]).

The following lemma comes from [5, Remark 1], which is a key to our proof.

**Lemma 3.1.** Let $G$ be a simple graph. If for every vertex $u$ of maximum degree there exists an edge $u - v$ such that $\Delta(G) - d(v) + 2$ is more than the number of vertices with maximum degree in $G$, then $\chi'(G) = \Delta(G)$.■

For the zero-divisor graph $\Gamma(R)$ of a finite commutative ring $R$, Akbari and Mohammadian have proved that $\chi'(\Gamma(R)) = \Delta(\Gamma(R))$ unless $\Gamma(R)$ is a complete graph of odd order [5, Theorem 3]. Note that if $R$ is reduced, then $\Gamma_N(R)$ is actually the zero-divisor graph $\Gamma(R)$. Furthermore, if $R$ is finite, then by the Wedderburn-Artin Theorem, $R$ is isomorphic to the direct product of finitely many fields. Thus $\Gamma_N(R)$ is not a complete graph. Therefore we conclude that if $R$ is a finite reduced commutative ring, then $\chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R))$. As usual, we use $K_n$ for a complete graph with $n$ vertices.

**Theorem 3.2.** Assume that $R$ is a finite nonreduced commutative ring. Then $\chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R))$ unless $R$ is a nilpotent ring with even order.

**Proof.** First assume that $R \neq N(R)$. It is clear that $\Delta(\Gamma_N(G)) = d(a) = |R^*| - 1$ for any $a \in N(R)^*$. Let $x \in R \setminus N(R)$. Since $x^2 \notin N(R)$, we have $x$ is not adjacent to any element of $x + N(R)^*$, so $d(x) \leq |R^*| - |N(R)^*| - 1$. It follows that $N(R)^*$ is the set of the vertices with maximal degree in $\Gamma_N(R)$ and so the number of vertices with maximal degree is $|N(R)^*|$. Now, for any vertex $a$ with maximum degree (i.e. $a \in N(R)^*$), $a$ is adjacent to any element $x$ in $R \setminus N(R)$ with $d(x) < |R^*| - |N(R)^*| < \Delta(\Gamma_N(G)) - |N(R)^*| + 2$. Hence $\chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R))$ by Lemma 3.1.

Now assume that $R = N(R)$. Then $R$ is a nilpotent ring and $\Gamma_N(G)$ is a complete graph. By the well-known result that $\chi'(K_n) = n - 1 = \Delta(K_n)$ if $n$ is even and $\chi'(K_n) = n = \Delta(K_n) + 1$ if $n$ is odd (see e.g. [15, Theorem 1.2]), the result follows immediately.■

Note that if $R$ is a finite commutative ring with identity, then it is not a nilpotent ring.

**Corollary 3.3.** Let $R$ is a finite commutative ring with identity. Then $\chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R))$.■

The question of when $\Gamma_N(R) \cong \Gamma_N(S)$ implies that $R \cong S$ is very interesting and this kind of question about the zero divisor graph of commutative rings
has been studied extensively in [14] and [5]. In [14], the authors have proved that if $R$ and $S$ are finite reduced rings which are not fields, then $\Gamma(R) \simeq \Gamma(S)$ if and only if $R \simeq S$. Akbari and Mohammadian generalized this result in [5] and proved that if $R$ is a finite reduced ring which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or to $\mathbb{Z}_6$ and $S$ is a ring such that $\Gamma(R) \simeq \Gamma(S)$, then $R \simeq S$. Analogously, we determine when $\Gamma_N(R) \simeq \Gamma_N(S)$ implies that $R \simeq S$. In the rest of this paper, we assume that all rings have identity.

Remark 5. (a) Assume that $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S$ is a nonreduced commutative ring. If $\Gamma_N(R) \simeq \Gamma_N(S)$, then by Remark 3, $|S| = 3$. Thus $S$ is a null ring of order 3.

(b) Assume that $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$ and $S$ is a nonreduced commutative ring. If $\Gamma_N(R) \simeq \Gamma_N(S)$, then by Remark 2, $|S| = 4$. Since $\Gamma_N(S)$ is a star graph, by Example 1, $S \simeq \mathbb{Z}_4$, or $S \simeq \mathbb{Z}_2[x]/(x^2)$.

Theorem 3.4. Let $R$ be a finite reduced commutative ring and $S$ be a commutative ring which is not an integral domain. If $\Gamma_N(R) \simeq \Gamma_N(S)$, then $R \simeq S$, unless $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $S$ is a null ring of order 3, or $R \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$ and $S$ is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

Proof. Assume that $R$ is a finite reduced commutative ring. If $S$ is also reduced, then by [5, Theorem 5], we are done. Now we assume that $S$ is nonreduced. Since $\Gamma_N(S)$ is finite, by Remark 3, $S$ is finite. Since $S$ is nonreduced, there exists a vertex which is adjacent to every other vertex in $\Gamma_N(S)$, and so does in $\Gamma_N(R)$. Since $R$ is reduced, $\Gamma_N(R)$ is actually the zero-divisor graph $\Gamma(R)$ of $R$. Note that $R$ is not a local ring, thus by [2, Corollary 2.7], we have $R \simeq \mathbb{Z}_2 \times F$, where $F$ is a finite field. Since $\Gamma(\mathbb{Z}_2 \times F)$ is a star graph, by [2, Lemma 2.12], we have that $|F| \leq 3$ and hence $F \simeq \mathbb{Z}_2$ or $\mathbb{Z}_3$. Thus by Remark 5, we complete the proof.

Theorem 3.5. Assume that $R$ and $S$ are finite nonreduced commutative rings such that $\Gamma_N(R) \simeq \Gamma_N(S)$. Then $|R| = |S|$ and $|N(R)| = |N(S)|$.

Proof. From the proof of Theorem 3.2, the maximal degree of $\Gamma_N(R)$ is $|R| - 1$, and the number of vertices with maximal degree in $\Gamma_N(R)$ is $|N(R)| - 1$. Thus the result follows immediately.

Corollary 3.6. Assume that $R$ and $S$ are finite nonreduced commutative rings with $N(R)$ of prime order. Then $\Gamma_N(R) \simeq \Gamma_N(S)$ if and only if $R \simeq S$.■
Corollary 3.7. Assume that $R$ and $S$ are finite nonreduced commutative principle ideal rings. Then $\Gamma_N(R) \cong \Gamma_N(S)$ if and only if $R \simeq S$.

References


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