Endomorphisms of Continuous Modules with Some Chain Conditions

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Dedicated to Professor A. Kaidi on his 60th birthday

Abstract. In this paper, we give some properties of the Jacobson radical of the endomorphism ring of continuous modules under various chain conditions. Namely, (i) if $M$ is a continuous module over a ring with ACC on annihilators of $M$, then the Jacobson radical of $\text{End}_R(M)$ is locally nil and annihilates some nonzero fully invariant submodule of $M$. (ii) In case that $M$ is indecomposable and the base ring is semiartinian hereditary, then the endomorphism ring is a division ring.

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Introduction

Continuous modules and their endomorphisms are subjects of intensive study in module theory, see for example [8], [2], and the bibliography therein. In particular, it was shown in [2] that every endomorphism of such module is clean, that it can be decomposed in a sum of a projection and an isomorphism. On the other hand, many authors (see [9], [12]) studied continuous rings with various chain conditions and gave interesting structure theorems.

We devote the present work, to the study of the endomorphism ring of a continuous module, under various chain conditions. In the first section, the base ring is supposed to satisfying ACC on annihilators of $M$. Here we show that the Jacobson radical of $\text{End}_R(M)$ is locally nil, and that $M$ contains a fully invariant submodule annihilated by $J(\text{End}_R(M))$. We also derive some consequences of these facts.
In section 2, we consider the case when the module is indecomposable. We show that, if the base ring is semiartinian hereditary, then the endomorphism ring of such a module is a division ring, generalizing many results in [5].

In all this work, the modules considered are unital left modules. For the background of module and ring theory, one may consult [1], [3], [7], [10] or [11]. If $R$ is any ring, we denote $J(R)$ the Jacobson radical of $R$, and $\text{Soc}(R)$ its left socle.

1. Continuous Modules under Chain Conditions on Annihilators

Recall that an $R$ module $M$ is said to be a continuous module if (C1) every submodule of $M$ is essential in direct summand of $M$, and (C2) every submodule of $M$ which is isomorphic to a direct summand is itself a direct summand.

In the sequel, we shall use the following chain condition:

**Definition.** (i) Let $M$ be an $R$-module. We note $\text{Ann}(M)$, the set of annihilators $\text{Ann}(x)$, where $x \in M$ and $\text{Ann}(x) = \{a \in R : ax = 0\}$. We say that $\text{Ann}(M)$ satisfies ACC (Ascending Chain Condition), if every nonempty subset of $\text{Ann}(M)$ has a maximal member.

It is clear that if $R$ is left noetherian, then $\text{Ann}(M)$ satisfies ACC for every $R$-module $M$.

(ii) An endomorphism $u$ of $M$ is said to be locally nilpotent, if its restriction on every finitely generated submodule is nilpotent. A subset $S$ of endomorphisms is said to be locally nil, if every element of $S$ is locally nilpotent.

(iii) Recall also that a submodule $N$ of a module $M$ is said to be fully invariant if $u(N) \subset N$ for every $u \in \text{End}_R(M)$.

**Theorem 1.** Let $M$ be a nonzero continuous module over a ring $R$, such that $\text{Ann}(M)$ satisfies ACC, Let $J(\text{End}_R(M))$ be the Jacobson radical of $\text{End}_R(M)$, and $N = \{x \in M : u(x) = 0, \text{ for every } u \in J(\text{End}_R(M))\}$, then:

(i) $N$ is a nonzero fully invariant submodule of $M$.

(ii) $J(\text{End}_R(M))$ is the largest locally nil ideal of $\text{End}_R(M)$.
(iii) If \( M \) is quasi-injective, then \( J(\text{End}_R(M)) = \{ u \in \text{End}_R(M) : u(N) = 0 \} \), and \( \text{End}_R(M)/J(\text{End}_R(M)) \cong \text{End}_R(N) \) which is von Neumann regular.

Proof. (i) The fact that \( N \) is a fully invariant submodule is clear. It remains to show that \( N \neq 0 \). Since \( \text{Ann}(M) \) satisfies ACC, the set \( F = \{ \text{Ann}(x) : x \in M \setminus \{0\} \} \), has a maximal member, \( I = \text{Ann}(x) \), say. We are now going to show that \( x \in N \), that is \( u(x) = 0 \), for every \( u \in J(\text{End}_R(M)) \). Otherwise, suppose there exists \( u \in J(\text{End}_R(M)) \) such that \( u(x) \neq 0 \). Since \( M \) is continuous, \( \text{Ker}_u \) is essential in \( M \), thus \( \text{Ker}_u \cap Rx \neq \{0\} \). Let \( ax \in \text{Ker}_u \) be nonzero. \( au(x) = u(ax) = 0 \). Hence \( a \in \text{Ann}(u(x)) \). But \( \text{Ann}(x) \subset \text{Ann}(u(x)) \). This containment is strict since \( au(x) = 0 \), and \( ax \neq 0 \). This contradict the maximality of \( \text{Ann}(x) \). Consequently, \( u(x) = 0 \).

(ii) Let \( u \in J(\text{End}_R(M)) \) we shall show that \( \forall x \in M \), there exists \( k \in \mathbb{N} \), such that \( u^k(x) = 0 \). Otherwise, there exists \( x \in M \) such that \( u^k(x) \neq 0 \) for all \( k \in \mathbb{N} \). Consider the set \( F = \{ \text{Ann}(u^k(x)) \} \). Then \( F \) has a maximal member, \( \text{Ann}(u^k(x)) \), say. As in (i), \( \text{Ann}(y) = \text{Ann}(u(y)) \) by maximality, where \( y = u^k(x) \). Now there exists \( a \in R \) such that \( ay \neq 0 \) and \( ay \in \text{Ker}_u \cap Ry \). Thus \( au(y) = u(ay) = 0 \), this means that \( a \in \text{Ann}(y) \). A contradiction. It follows that every \( u \in J(\text{End}_R(M)) \) is locally nilpotent. On the other hand, every locally nilpotent endomorphism is quasi-invertible, thus every locally nil ideal of \( \text{End}_R(M) \) is contained in \( J(\text{End}_R(M)) \).

(iii) Suppose that the module \( M \) is quasi-injective. Consider the ring morphism: \( \phi : \text{End}_R(M) \to \text{End}_R(N) \), defined by \( \phi(u) = u|_N \), where \( u|_N \) is the restriction of \( u \) to \( N \). \( \phi \) is surjective by the quasi-injectivity of \( M \). Now \( \text{Ker}_\phi = \{ u \in \text{End}_R(M) : u(N) = 0 \} = J(\text{End}_R(M)) \). Thus \( \text{End}_R(N) \cong \text{End}_R(M)/J(\text{End}_R(M)) \) and this is a von Neumann regular ring by the well-known Johnson and Wong theorem. ■

Remark. Note that if \( M \) is nonsingular, then \( N = M \). On the other hand, if we take \( R = \mathbb{Z} \), the ring of integers and \( M = \mathbb{Z}_p(\infty) \), the Prüfer group relative to a prime \( p \), then \( M \) is injective singular (\( \text{Sing}(M) = M \)), and \( N = S \), where \( S \) is the socle of \( \mathbb{Z}_p(\infty) \), which is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \).

Corollary 2. Let \( M \) be a nonzero indecomposable continuous \( R \)-module such that \( \text{Ann}(M) \) satisfies ACC. Then there exists a nonzero submodule \( N \) of \( M \) such that \( J(\text{End}_R(M)) = \{ u \in \text{End}_R(M) : u(N) = 0 \} \).

Proof. Let \( N \) be as in theorem 1. We have \( J(\text{End}_R(M))(N) = 0 \). On the other hand, if \( u(N) = 0 \), by using the fact that \( M \) is uniform, then \( \text{Ker}_u \) is essential. Thus \( u \in J(\text{End}_R(M)) \). ■
Corollary 3. Let $M$ a nonzero indecomposable quasi-injective module such that $\text{Ann}(M)$ satisfies ACC Then there exists a submodule $N$ of $M$ such that $\text{End}_R(N)$ is a division ring.

Proof. $\text{End}_R(N)$ is VNR and $N$ is uniform. ■

Now we derive to another kind of chain conditions.

Definition. A module is said to be Hopfian (co-Hopfian) provided that every surjective (resp. injective) endomorphism of $M$ is an isomorphism.

Following [6], an $R$-module $M$ is said to be strongly Hopfian (resp. strongly co-Hopfian), if for every endomorphism $u$ of $M$ the chain $\text{Ker}u \subset \text{Ker}u^2 \subset \ldots \text{Ker}u^n \subset \ldots$ (resp. $\text{Im}u \supset \text{Im}u^2 \supset \ldots \supset \text{Im}u^n \supset \ldots$), stabilizes.

Proposition 4. Let $M$ be a continuous strongly Hopfian module over a ring $R$. Then the Jacobson radical of $\text{End}_R(M)$ is nil.

Proof. Let $u \in J(\text{End}_R(M))$. Since $M$ is strongly Hopfian, then there exists $k \in \mathbb{N}$, such that $\text{Ker}u^k \cap \text{Im}u^k = \{0\}$. Since $\text{Ker}u^k$ is essential, $\text{Im}u^k = \{0\}$. Hence $u^k = 0$. ■

2. Endomorphisms of Indecomposable Continuous Modules

In this section, we study the endomorphism ring of an indecomposable continuous module. In particular, we give many instances where this ring is a division ring, (such modules are called fieldendo in [4]), which is the extreme case of Hopficity and co-Hopficity. By the way, we obtain many generalizations of the results given [5].

Definition. A module is said to be semi-Hopfian (semi-co-Hopfian) if every surjective (resp. injective) endomorphism of $M$ is right invertible (resp. left invertible).

First, we record the following easy fact:

Proposition 5. Let $M$ be an $R$ module, then the following assertions are equivalent:

(i) $M$ is indecomposable continuous.

(ii) $M$ is uniform and semi-co-Hopfian.
Proof. This is clear from the fact that condition (C2) is equivalent to the semi-co-Hopficity in case of indecomposable modules. ■

Recall that a ring $R$ is said to be semiartinian if every nonzero $R$-module has nonzero socle.

Proposition 6. Let $M$ be a module over a semiartinian ring $R$, and $u$ an endomorphism of $M$ such that $u(\text{soc}(M)) = 0$. Then $u$ is not surjective.

Proof. First case: Suppose that $\text{Ann}(M) = \{a \in R : aM = 0\} = 0$, that is $M$ is faithfull. Put $L = \text{Soc}(R)$, then $LM \subset \text{Soc}(M)$. But $u(L(M)) = 0$, thus $Lu(M) = 0$. This implies that $u(M) \neq M$, that is $u$ is non surjective.

General case: Let $I = \text{Ann}(M)$. Then $M$ is a faithfull $S$-module, where $S = R/I$. Moreover, $\text{End}_R(M) = \text{End}_S(M)$ and the socles of $M$ as $R$-module and as $S$-modules are equal. We can therefore apply the first case. ■

Corollary 7. Let $R$ be a semiartinian ring and $M$ a nonzero uniform module over $R$. Then $M$ is Hopfian.

Proof. If $u \in \text{End}_R(M)$ is not injective, then, since $M$ is uniform, $u(\text{soc}(M)) = 0$. Thus $u$ is not surjective.

Let $R$ be a ring. Then $R$ is said to be hereditary, provided that every submodule of a projective module is projective. Or equivalently, if every factor module of an injective module is injective.

Proposition 8. Let $M$ be a continuous module over a hereditary semiartinian ring. If $u \in J(\text{End}_R(M))$, then there exists a nonzero projection $p$ of $M$ such that $pu = 0$.

Proof. Let $u \in J(\text{End}_R(M))$ and $\tilde{u}$ some extension of $u$ to injective hull $\hat{M}$ of $M$. Then $\tilde{u}(\text{soc}(\hat{M})) = 0$. By Proposition 6, $\tilde{u}$ is not surjective. Since $R$ is hereditary, $\text{Im} \tilde{u}$, is an injective submodule of $\hat{M}$, and then a summand of $\hat{M}$. Consequently, there exists a projection $q$ of $\hat{M}$ such that $q\tilde{u} = 0$. But $M$ is continuous, thus $M$ is invariant under $q$. Now if we put $p = q|_M$, then $p$ is a nonzero projection of $M$, and $pu = 0$, as desired. ■

Corollary 9. Let $R$ be a ring which is a factor of a hereditary semiartinian ring, then for every indecomposable continuous $R$ module $M$, $\text{End}_R(M)$ is a
division ring.

Proof. By [5], Proposition 8, it suffices to show the result for semiartinian hereditary ring. If \( u \in J(\text{End}_R(M)) \), then by the preceding proposition, there exists a nonzero projection \( p \) of \( M \) such that \( pu = 0 \). But \( M \) is indecomposable, thus \( p = I_M \) and \( u = 0 \). This means that \( J(\text{End}_R(M)) = 0 \). ■

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References

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