A Note on Finitely Generated Multiplication Semimodules over Commutative Semirings

S. Ebrahimi Atani and M. Shajari Kohan

Department of Mathematics
University of Guilan
P.O. Box 1914 Rasht Iran

Abstract

This paper deals with finitely generated multiplication semimodules, and the concept of the $M$-radical of a subsemomodule $N$ of an $R$-semimodule $M$ is discussed ($R$ is a commutative semiring with identity and $M$ is a unitary $R$-semimodule). The results for multiplication semimodules are generalization of corresponding results for multiplication modules.

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1 Introduction

The study in semirings has been carried out by several authors since there are numerous applications of semirings in various branches of mathematics and computer science (see [6, 7]). The main part of this paper is devoted to extending some basic results of the notion finitely generated multiplication from the theory of modules to theory of semimodules (see Section 2, 3).

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring we mean an algebraic system $R = (R, +, .)$ such that $(R, +)$ and $(R, .)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exists $0 \in R$ such that $r + 0 = r$ and $r.0 = 0.r = 0$ for all $r \in R$. Throughout this paper let

\[\text{1ebrahimiatani@gmail.com}\]
$R$ be a commutative semiring. A (left) semimodule $M$ over a semiring $R$ is a commutative additive semigroup which has a zero element, together a mapping from $R \times M$ into $M$ (sending $(r, m)$ to $rm$) such that $(r + s)m = rm + sm$, $r(m + p) = rm + rp$, $r(sm) = (rs)m$ and $0m = r0_M = 0_M$ for all $m, p \in M$ and $r, s \in R$. Let $M$ be a semimodule over the semiring $R$, and let $N$ be a subset of $M$. We say that $N$ is a subsemimodule of $M$, or an $R$-subsemimodule of $M$, precisely when $N$ is itself an $R$-semimodule with respect to the operations for $M$ (so $0_M \in N$). It is easy to see that if $r \in R$, then $rM = \{rm : m \in M\}$ is a subsemimodule of $M$. The semiring $R$ is considered to be also a semimodule over itself. In this case, the subsemimodules of $R$ are called ideals of $R$. Let $M$ be a semimodule over a semiring $R$. A subtractive subsemimodule (= $k$-subsemimodule) $N$ is a subsemimodule of $M$ such that if $x, x + y \in N$, then $y \in N$ (so $\{0_M\}$ is a $k$-subsemimodule of $M$). A prime subsemimodule (resp. primary subsemimodule) of $M$ is a proper subsemimodule $P$ of $M$ in which $x \in P$ or $rM \subseteq P$ (resp. $x \in P$ or $r^nM \subseteq P$ for some positive integer $n$) whenever $rx \in N$. An $R$-semimodule $M$ is called a multiplication semimodule whenever $N$ is a subsemimodule of $M$, then there exists an ideal $I$ of $R$ such that $N = IM$ [2]. In this case we can take $I = (N : M) = \{r \in R : rM \subseteq N\}$.

We define $k$-ideals and prime $k$-ideals of a semiring $R$ in a similar fashion.

## 2 Properties of multiplication semimodules

In this section we list some basic properties concerning finitely generated multiplication semimodules over a semiring.

**Remark 2.1** (1) Let $R$ be a semiring. We define the Jacobson radical of $R$, denoted by $\text{Jac}(R)$, to be the intersection of all the maximal $k$-ideals of $R$. Then by [5, Lemma 2], the Jacobson radical of $R$ always exists and by [3, Lemma 2.12], it is a $k$-ideal of $R$. A non-zero element $a$ of $R$ is said to be semi-unit in $R$ if there exist $r, s \in R$ such that $1 + ra = sa$.

(2) An proper ideal $I$ of a semiring $R$ is said to be a strong ideal, if for each $a \in I$ there exists $b \in I$ such that $a + b = 0$ (note that finite sum of strong ideals of $R$ is a strong ideal of $R$).

(3) Let $R = \{0, 1, 2, \ldots, 20\}$, and define $a + b = \max\{a, b\}$, $a \cdot b = \min\{a, b\}$ for each $a, b \in R$. Then $(R, +, \cdot)$ is easily checked to be a commutative semiring with 20 as identity. Let $J_4$ denote the ring integer modulo 4. Let $J_4 \oplus R = \{(a, b) : a \in J_4, b \in R\}$ denote the direct sum of semirings $J_4$ and $R$. Then $J_4 \oplus R$ is a commutative semiring. An inspection will show that $I_0 = \{(a, 0) :$
Let \( M \) be a finitely generated semimodule over a semiring \( R \), and let \( J \) be a strong ideal in \( R \) such that \( JM = M \). Then \( (1 + t)M = 0 \) for some \( t \in J \). In particular, \( R = J + \text{ann}(M) \).

**Proof.** Let \( M = \langle m_1, m_2, ..., m_n \rangle \). We use induction on \( n \). Consider first the case in which \( n = 1 \). Here we have \( \langle m_1 \rangle = J < m_1 \rangle >. \) So \( m_1 = sm_1 \) for some \( s \in J \); hence there is an element \( s' \in J \) such that \( (1 + s')m_1 = sm_1 + s'm_1 = 0 \). It follows that \( (1 + s')M = 0 \). We now turn to the inductive step. Assume, inductively, that \( n = k + 1 \), where \( k \geq 1 \), and that the result has been proved in the case where \( n = k \). Then we must have \( (1 + a)(1 + b)M = (1 + a + b + ab)(< m_1, ..., m_k > + < m_{k+1} >) = 0 \) for some \( a, b \in J \), so \( (1 + t)M = 0 \), where \( a + b + ab = t \in J \). Finally, there exist \( t \in J \) and \( c \in \text{ann}(M) \) such that \( 1 + t = c \), so \( r + rt = rc \) for all \( r \in R \). By assumption, there is an element \( d \in J \) such that \( rt + d = 0 \); hence \( r = rc + d \in J + \text{ann}(M) \), as required. \( \square \)

**Proposition 2.3** If \( M \) is a finitely generated semimodule over a semiring \( R \), \( P \) is a strong prime \( k \)-ideal of \( R \) containing \( \text{ann}(M) \), and \( I \) is an ideal of \( R \) such that \( IM \subseteq PM \), then \( I \subseteq P \). In particular, \( (PM : M) = P \).

**Proof.** Assume that \( M = Rm_1 + ... + Rm_n \) and let \( r \in I \). First we show that there is an element \( d = r^n + c \in R \) such that \( dM = 0 \), where \( c \in P \). We use induction on \( n \). Consider first the case in which \( n = 1 \). Here we have \( Im_1 \subseteq Pm_1 \). By assumption, there exist \( p, p' \in P \) such that \( p + p' = 0 \) and \( rm_1 = pm_1 \), so \( (r + p')M = 0 \). We now turn to the inductive step. Assume, inductively, that \( n = k + 1 \), where \( k \geq 1 \), and that the result has been proved in the case where \( n = k \). Therefore, there are elements \( q, q' \in P \) such that \( (r + q)(r^k + q')M = (r^{k+1} + rq' + r^kq + qq')(< m_1, ..., m_k > + < m_{k+1} >) = (r^{k+1} + c)M = 0 \), where \( c = rq' + r^kq + qq' \in P \). Thus \( dM = 0 \) and so \( d \in \text{ann}(M) \subseteq P \). Therefore, \( d \in P \) and \( r \in P \). Finally, Clearly,
Let \( P \subseteq (PM : M) \). For the other inclusion, since \((PM : M)M \subseteq PM\), we get \((PM : M) \subseteq P\), and hence we have equality. \( \square \)

**Proposition 2.4** Let \( R \) be a semiring. An \( R \)-semimodule \( M \) is multiplication semimodule if and only if for each \( x \) in \( M \) there exists an ideal \( J \) of \( R \) such that \( Rx = JM \).

**Proof.** The necessity is clear. For the sufficiency, suppose that for each \( x \in M \) there exists an ideal \( J_x \) such that \( Rx = J_x M \). Let \( N \) be a subsemimodule of \( M \). For each \( x \in N \) there exists an ideal \( J_x \) such that \( Rx = J_x M \). Let \( J = \sum_{x \in N} J_x \). Then \( N = \sum_{x \in N} Rx = \sum_{x \in N} J_x M = JM \). It follows that \( M \) is a multiplication semimodule. \( \square \)

**Theorem 2.5** Let \( M \) be a multiplication semimodule over a semiring \( R \). If \( N \) is a finitely generated subsemimodule of \( M \), then there exists a finitely generated ideal \( I \) of \( R \) such that \( N = IM \).

**Proof.** Let \( N = \langle x_1, x_2, ..., x_n \rangle \). by assumption, we have \( N = (N : M)M \). Therefore, there exists \( a_{i,j} \in (N : M) \) and \( y_{i,j} \in M \) such that \( x_i = a_{i,1}y_{i,1} + ... + a_{i,s}y_{i,s} \) for \( i = 1, 2, ..., n \) and \( j = 1, 2, ..., s \). Let \( I \) be an ideal of \( R \) generated by \( \{a_{1,1}, ..., a_{n,s}\} \). It is easy to see that \( I \subseteq (N : M) \) and \( IM \subseteq (N : M)M \). On the other hand, since for each \( i, x_i \in IM \), we must have \( N \subseteq IM \). Thus \( N \subseteq IM \subseteq (N : M)M \subseteq N \). Hence \( N = IM \) and \( I \) is finitely generated. \( \square \)

**Theorem 2.6** Let \( M \) be a multiplication semimodule over a semiring \( R \) and let \( J \) be a strong ideal of \( R \) contained in the Jacobson radical of \( R \). Then \( M = JM \) implies \( M = 0 \).

**Proof.** Given \( x \in M \), there is an ideal \( K \) of \( R \) such that \( Rx = KM \). Thus \( Rx = KM = KJM = JKM = Jx \). Therefore \( x = bx \) for some \( b \in J \). There is an element \( c \in J \) such that \( c + b = 0 \), so \((1 + c)x = 0\). Since \( 1 + c \) is a semi-unit by [4, Lemma 3.4], we must have \( 1 + (1 + c)t = (1 + c)s \) for some \( t, s \in R \); hence \( x = 0 \). \( \square \)

## 3 Radicals of subsemimodules

Let \( R \) be a semiring. We define the \( M \)-radical of a subsemimodule \( N \) of an \( R \)-semimodule \( M \) to be intersection of all prime semimodules of \( M \) containing \( N \) (denoted \( rad(N) \)). The proof of the following lemma is straightforward.
Lemma 3.1 Let \( R \) be a semiring, \( M \) an \( R \)-semimodule and \( N \) an \( R \)-subsemimodule of \( M \). Then the following assertion are equivalent.

(i) \( N \) is a prime subsemimodule of \( M \).

(ii) If whenever \( IT \subseteq N \) (with \( I \) an ideal of \( R \) and \( T \) a subsemimodule of \( M \)) implies that \( I \subseteq (N :_R M) \) or \( T \subseteq N \).

Theorem 3.2 If \( M \) is a finitely generated multiplication semimodule over a semiring \( R \), \( P \) is a strong prime \( k \)-ideal of \( R \) containing \( \text{ann}(M) \), then \( PM \) is a prime subsemimodule of \( M \).

Proof. Note that \( PM \neq M \). Otherwise by Lemma 2.2, there exists \( p \in P \) such that \( 1 + p \in \text{ann}(M) \subseteq P \), which is a contradiction. Suppose that \( I \) is an ideal of \( R \) and \( N \) is a subsemimodule of \( M \) such that \( IN \subseteq PM \). Since \( M \) is multiplication module, \( N = JM \) for some ideal \( J \) of \( R \). Then \( IN = IJM \subseteq PM \). By Proposition 2.3, \( IJ \subseteq P \), so \( I \subseteq P \) or \( J \subseteq P \); hence \( I \subseteq P = (PM : M) \) by Proposition 2.3 or \( J \subseteq P \) then \( JM = N \subseteq PM \). By Lemma 3.1, \( PM \) is a prime subsemimodule of \( M \). \( \square \)

Lemma 3.3 Let \( R \) be a semiring, \( M \) an \( R \)-semimodule and \( N \) a subsemimodule of \( M \). Then the following hold:

(i) If \( N \) is prime subsemimodule of \( M \), then \( (N : M) \) is a prime \( k \)-ideal of \( R \).

(ii) If \( N \) is a primary subsemimodule of \( M \), then \( \sqrt{(N : M)} \) is a prime ideal of \( R \) containing \( \text{ann}(M) \).

Proof. (i) Since \( N \) is a proper subsemimodule, we must have \( (N :_R M) \neq R \). To show that \( (N : M) \) is prime ideal, suppose that \( rs \in (N : M) \), so that \( rsM \subseteq N \). Either \( sM \subseteq N \) or \( sa \in M - N \) for some \( a \in M \). In the later case since \( N \) is a prime semimodule and \( r(sa) \in N \), we must have \( rM \subseteq N \). Thus \( r \in (N : M) \) or \( s \in (N : M) \) and \( (N : M) \) is prime.

(ii) Let \( a, b \in R \) such that \( ab \in \sqrt{(N : M)} \), \( b \notin \sqrt{(N : M)} \). Then \( a^n(b^m)M \subseteq N \) and \( b^nM \notin N \) for some \( m \in M - N \) and positive integer \( n \). Since \( N \) is primary and \( a^n(b^m)M \subseteq N \), we must have \( a \in \sqrt{(N : M)} \), as needed. \( \square \)

Proposition 3.4 Assume that \( M \) is a semimodule over a semiring \( R \) and let \( N \) be a subsemimodule of \( M \). Then \( \sqrt{(N : M)}M \subseteq \text{rad}(N) \).

Proof. If \( \text{rad}(N) = M \) the result is immediate. Otherwise, if \( K \) is any prime subsemimodule of \( M \) containing \( N \), we have \( (N : M) \subseteq (K : M) \). It follows from Lemma 3.3 that \( \sqrt{(N : M)} \subseteq (K : M) \) and hence \( \sqrt{(N : M)}M \subseteq (K : M)M \subseteq K \). Since \( K \) is an arbitrary prime subsemimodule containing \( N \), we have \( \sqrt{(N : M)}M \subseteq \text{rad}(N) \). \( \square \)
Theorem 3.5 Let $M$ be a finitely generated multiplication semimodule over a semiring $R$ and let $N$ be a subsemimodule of $M$ such that $(N : M)$ is a strong ideal of $R$. Then $\sqrt{(N : M)M} = \text{rad}(N)$.

Proof. By Proposition 3.4, $\sqrt{(N : M)M} \subseteq \text{rad}(N)$. Since $M$ is a multiplication semimodule, we must have $\text{rad}(N) = (\text{rad}(N) : M)M$. It is enough to show that $(\text{rad}(N) : M) \subseteq \sqrt{(N : M)}$. Since $(N : M)$ is a strong $k$-ideal, there is a strong prime $k$-ideal $P$ of $R$ such that $(N : M) \subseteq P$. As $P$ is a strong prime $k$-ideal containing $\text{ann}(M) \subseteq (N : M)$, then $PM$ is a prime subsemimodule of $M$ containing $N = (N : M)M$ by Theorem 3.2. Hence $(\text{rad}(N) : M)M = PM$, so that $(\text{rad}(N) : M) \subseteq P$ by Proposition 2.3. Therefore, $(\text{rad}(N) : M) \subseteq \sqrt{(N : M)}$. $\square$

Corollary 3.6 Let $M$ be a finitely generated multiplication semimodule over a semiring $R$ and let $N$ be a primary subsemimodule of $M$ such that $\sqrt{(N : M)}$ is a strong ideal of $R$. Then $\text{rad}(N)$ is a prime subsemimodule of $M$.

Proof. By Lemma 3.3, $\sqrt{(N : M)}$ is a prime ideal containing $\text{ann}(M)$. Since every subideal of a strong ideal is a strong ideal, $\text{rad}(N) = \sqrt{(N : M)}M$ is a prime subsemimodule of $M$ by Theorem 3.5 and Theorem 3.2. $\square$

Let $M$ be a semimodule over a semiring $R$. The set of all strong prime $k$-subsemimodules of $M$ is called $k$-spectrum of $M$ and denoted by $\text{Spec}_k(M)$.

Proposition 3.7 Let $R$ be a semiring, $I$ a strong $Q$-ideal of $R$ and $P$ a $k$-ideal of $R$ with $I \subseteq P$. Then $P$ is a strong prime $k$-ideal of $R$ if and only if $P/I$ is a strong prime $k$-ideal of $R/I$.

Proof. By [3, Theorem 2.5], it suffices to show that $P$ is a strong ideal of $R$ if and only if $P/I$ is a strong ideal of $R/I$.

First suppose that $P/I$ is a strong ideal of $R/I$. To see that $P$ is strong, assume that $a \in P$. Then $a = q + c \in q + I$ for some $c \in I$. Suppose that $q_0$ is the unique element in $Q$ such that $q_0 + I$ is the zero in $R/I$. Since $P$ is a $k$-ideal of $R$, we must have $q \in P \cap Q$, so $q + I \in P/I$. By assumption, $(q + I) \oplus (q' + I) = q_0 + I = I$ for some $q' + I \in P/I$, where $q + q' + I \subseteq I$; hence $q + q' + e = f$ for some $e, f \in I$. As $f + c \in I$, there is an element $f' \in I$ such that $f + c + f' = 0$, so $a + (q' + c + f') = c + f + f' = 0$; hence $P$ is a strong ideal. Conversely, suppose that $q + I \in P/I$, where $q \in Q \cap P$ by [3, Proposition 2.2], so $q + d = 0$ for some $d \in P$. Then there exists $q_1 \in Q$ such that $d = q_1 + g \in q_1 + I$, where $g \in I$ and hence $q + q_1 + g = 0$. An inspection will show that $(q + I) \oplus (q_1 + I) = q_0 + I$. Thus $P/I$ is a strong ideal of $R/I$. $\square$
Theorem 3.8 Let $M$ be a finitely generated multiplication semimodule over a semiring $R$ with $\text{ann}(M)$ is a strong $Q$-ideal, and let $(N : M)$ be a strong ideal of $R$ for any prime $k$-subsemimodule $N$ of $M$. Then there exists a bijection $\varphi : \text{Spec}_k(M) \to \text{Spec}_k(R/\text{ann}(M))$.

Proof. For any prime $k$-subsemimodule $N$ of $M$, we define $\varphi(N) = (N : M)/\text{ann}(M)$. Note that $(N : M)$ is a strong prime $k$-ideal of $R$ by Lemma 3.3 and $(N : M)/\text{ann}(M)$ is a strong prime $k$-ideal of $R/\text{ann}(M)$ by Proposition 3.7. Let $P/\text{ann}(M)$ be any strong prime $k$-ideal of $R/\text{ann}(M)$. Then by Theorem 3.2, $PM$ is a prime subsemimodule of $M$ and $\varphi(PM) = (PM : M)/\text{ann}(M) = P/\text{ann}(M)$ by Proposition 2.3 and so $\varphi$ is onto. Suppose that $\varphi(N) = \varphi(N')$. Then $(N : M)/\text{ann}(M) = (N' : M)/\text{ann}(M)$ and so by [3, Lemma 2.13], $(N : M) = (N' : M)$. Since $M$ is a multiplication semimodule, $N = (N : M)M = (N' : M)M = N'$, as required. □

Corollary 3.9 Let $M$ be a faithful finitely generated multiplication semimodule over a semiring $R$, and let $(N : M)$ be a strong ideal of $R$ for any prime subsemimodule $N$ of $M$. Then there exists one to one correspondence between $\text{Spec}_k(M)$ and $\text{Spec}_k(R)$.

Proof. Note that $\text{ann}(M) = 0$ is a strong $Q$-ideal of $R$ with $Q = R$. Now apply Theorem 3.8. □

References


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