Integral Expression of Dirichlet L-Series

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Abstract

In this paper, we study the Mahler measure of a family of reciprocal polynomials depending of a parameter $l$. These polynomials do not vanish on the torus and define curves of genus 1 except for the singular values of the parameter. Using Mahler measure, we prove that Picard-Fuchs equation associated to this family admits explicit solutions at $\pm\infty$ and express certain Dirichlet $L$-series $L(\chi, 2)$ as integrals of functions related to $F\left(\frac{1}{2}, \frac{1}{2}, 1; z\right)$.

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1 Introduction

In a previous paper Villegas [4] proved the formula

$$\frac{1}{\pi} \int_0^1 (K - \frac{\pi}{2}) \frac{dk}{k} = \log 2 - d_4,$$

where

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

and

$$d_4 := \frac{4^{3/2}}{4\pi} L(\chi_{-4}, 2) = L'(\chi_{-4}, -1) = \frac{2G}{\pi}.$$ 

The constant $G$ is the Catalan’s constant

$$G = L(\chi_{-4}, 2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$
The formula
\[ \frac{1}{\pi} \int_0^1 Kdk = d_4 \]  
(2)
can be proved similarly. The formulae (1) and (2) are also mentioned without demonstration in the book of Byrd and Friedman [3]. There is also another proof of the second one in Borwein’s book [2]. In a recent paper [1], I proved the formulae

\[ L'(\chi_{-8}, -1) = 2 \log(\sqrt{2} + 1) + \log 2 - \frac{4\sqrt{2}}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{2 - k^2}} dk. \]

and

\[ L'(\chi_{-8}, -1) = \frac{2\sqrt{2}}{\pi} \int_0^1 K \left( \frac{u^2 + 1}{2} \right) du. \]

In this article, we study the logarithmic Mahler measure of a family of elliptic curves given by reciprocal polynomials depending of a parameter \( l \):

\[ P_l(x, y) = y^2(x + 1)^2 + y(x^2 + (l + 2)x + 1) + (x + 1)^2, \quad l \in R. \]

We prove that the derivative of the Mahler measure of \( P_l(x, y) \) with respect to the parameter is a period of the curve. An integration with respect to \( k \), the knowledge of the Mahler measure for the singular values of the parameter and a passage to the limit allows to get the formulae :

\[ L'(\chi_{-4}, -1) = \frac{3}{4\pi} \int_0^1 K \left( \sqrt{\frac{u^2 + 3}{4}} \right) du. \]

\[ L'(\chi_{-4}, -1) = \frac{3}{4} \log(3) - \frac{3}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} dk. \]

\[ L'(\chi_{-3}, -1) = \frac{3\sqrt{3}}{10\pi} \int_0^1 K \left( \sqrt{\frac{3t^2 + 1}{4}} \right) dt. \]

\[ L'(\chi_{-3}, -1) = \frac{-6}{5\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - k^2}} dk + \frac{3}{10} \log(2 + \sqrt{3}). \]
2 Definitions and Known results

If $P$ denotes a Laurent polynomial in $n$ variables, $P \in \mathbb{C}[X_1^\pm, \cdots, X_n^\pm]$ and $T^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n; |x_1| = \cdots = |x_n| = 1\}$ is the $n$ torus, then the logarithmic Mahler measure is defined by

$$m(P) := \frac{1}{(2\pi i)^n} \int_{T^n} \log \left| P(x_1, \cdots, x_n, \frac{1}{x_1}, \cdots, \frac{1}{x_n}) \right| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and the Mahler measure $M(P)$ by

$$M(P) := \exp(m(P)).$$

If $n = 1$ and $P(x) = a_0 \prod_{j=1}^d (x - \alpha_j)$, applying Jensen’s formula, we get

$$\int_0^1 \log |P(\exp(2i\pi t))| \, dt = \log |a_0| + \log \prod_{j=1}^d (\max(|\alpha_j|, 1),$$

i.e.

$$M(P) = |a_0| \prod_{j=1}^d (\max(|\alpha_j|, 1).$$

We recall the formula deduced from the functional equation of Dirichlet’s series,

$$L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2),$$

where $\chi_{-f}(n) = \left(\frac{-f}{n}\right)$ is the real odd Dirichlet character with conductor $f$.

**Definition 2.1.** We define the hypergeometric series by

$$F(z) = F\left(\frac{1}{2}, \frac{1}{2}, 1; z\right) = 1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1 \times 3}{2 \times 4}\right)^2 z^2 + \cdots$$

It verifies

$$K(k) = \frac{\pi}{2} F(k^2)$$

where

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \, dx$$
3 Proof of the main theorem

Theorem 3.1. Consider the family
\[ P_l(x, y) = y^2(x + 1)^2 + y(x^2 + (l + 2)x + 1) + (x + 1)^2, \quad l \in \mathbb{R}. \]

In what follows, we use the notations
\[ m(P_l) = m(l) \quad \text{and} \quad m'(l) = \frac{d}{dl} m(l). \]

1. If \( l > 4 \), then
\[ m'(l) = \frac{2}{\pi \sqrt{l^2 + 12l}} K \left( \frac{16}{l + 12} \right) \]
is a solution of Picard-Fuchs differential equation of the family \( P_l(x, y) \) in a vicinity of \(+\infty\).

2. If \( l > 4 \), then
\[ m(l) = -\frac{8}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k \sqrt{4 - 3k^2}} dk - 2 \log 2 + 2 \log \left( \sqrt{l + 12} + \sqrt{l} \right). \]

3. We have
\[ m(4) = 2 \log(3) - \frac{8}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k \sqrt{4 - 3k^2}} dk. \]

4. If \( l < -12 \), then
\[ m'(l) = -\frac{2}{\pi} \frac{1}{\sqrt{l^2 - 4l}} K \left( \frac{16}{4 - l} \right) \]
is a solution of Picard-Fuchs differential equation of the family \( P_l(x, y) \) in a vicinity of \(-\infty\).

5. If \( l < -12 \), then
\[ m(l) = -\frac{8}{\pi} \int_1^k \frac{K - \frac{\pi}{2}}{k \sqrt{4 - k^2}} dk + 2 \log \frac{\sqrt{4 - l} + \sqrt{-l}}{2}. \]

6. We have
\[ m(-12) = -\frac{8}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k \sqrt{4 - k^2}} dk + 2 \log(2 + \sqrt{3}). \]
7. If $0 < l < 4$, then
\[
m'(l) = \frac{1}{2\pi \sqrt{l}} K \left( \sqrt{\frac{l + 12}{16}} \right).
\]

8. We have
\[
m(4) = \frac{2}{\pi} \int_{0}^{1} K \left( \sqrt{\frac{u^2 + 3}{4}} \right) du.
\]

9. If $-12 < l < 0$, then
\[
m'(l) = \frac{-1}{2\pi \sqrt{-l}} K \left( \sqrt{\frac{4 - l}{16}} \right).
\]

10. We have
\[
m(-12) = \frac{2\sqrt{3}}{\pi} \int_{0}^{1} K \left( \sqrt{\frac{3t^2 + 1}{4}} \right) dt.
\]

Proof. The Mahler measure of $P_l$ is given by
\[
m(l) = \frac{1}{(2i\pi)^2} \int_{|x|=1} \int_{|y|=1} \log \left| (y + \frac{1}{y})x^2 + 2(y + l + \frac{1}{y})x + y + \frac{1}{y} \right| \frac{dx}{x} \frac{dy}{y}.
\]

We set $y = \exp(is)$; so
\[
m(l) = \frac{1}{\pi} \int_{0}^{\pi} f(s) ds
\]

with
\[
f(s) = \frac{1}{2i\pi} \int_{|x|=1} \log \left| x^2 (2 \cos s + 1) + x (4 \cos s + l + 2) + (2 \cos s + 1) \right| \frac{dx}{x}
\]

and
\[
P_{l,s}(x) = x^2 (2 \cos s + 1) + x (4 \cos s + l + 2) + (2 \cos s + 1).
\]
The discriminant $\Delta$ de $P_{l,s}$ is equal to $\Delta = l (l + 8 \cos s + 4)$.; as a result, using Jensen’s formula, we get
• if $\Delta > 0$, then
  \[ f(s) = \log |(2 \cos s + 1) \max (|X_1|, |X_2|)| \]
  where $X_1, X_2$ are roots of $P_{l,s}(x)$,

• if $\Delta \leq 0$, then
  \[ f(s) = \log |2 \cos s + 1| \]

1. If $l > 4$ then $\Delta > 0$, so, using Jensen formula, we have
  \[ m(l) = \frac{1}{\pi} \int_0^\pi \log |4 \cos s + l + 2 + \sqrt{\Delta}| \, ds. \]

By differentiating with respect to $l$, we obtain
  \[ m'(l) = \frac{1}{\pi} \int_0^\pi \frac{ds}{\sqrt{l(l + 8 \cos s + 4)}}. \]

Let $X = \sin \frac{s}{2}$ and $k^2 = \frac{16}{l + 12}$, then
  \[ m'(l) = \frac{2}{\pi \sqrt{l^2 + 12l}} K \left( \sqrt{\frac{16}{l + 12}} \right). \]

We put $T = \frac{1}{l}$ so we get $m'(l) = T f(T)$ and $f(T)$ admits entire series expansion in a vicinity of 0.

Let $K$ be the set of complex numbers $l$ for which $P_l(x, y)$ vanishes on the torus $T^2 = \{(x, y), |x| = |y| = 1\}$.

For $l \notin K$, $\log P_l(x, y)$ is an analytic function of $l$ and if we define
  \[ \tilde{m}(P_l) = \frac{1}{(2\pi i)^2} \int_{|x|=1} \int_{|y|=1} \log P_l(x, y) \frac{dx}{x} \frac{dy}{y}, \]
  then
  \[ m(P_l) = \Re(\tilde{m}(P_l)), \]
  so
  \[ \frac{d\tilde{m}(P_l)}{dl} = -\frac{1}{4\pi^2} \int_{T^2} P'_l(x, y) \, dx \, dy. \]

For the previous family, we get
  \[ \frac{d\tilde{m}(P_l)}{dl} = \]
\[-\frac{1}{4\pi^2} \int_{T^2} \frac{1}{(xy + \frac{1}{xy}) + (y + \frac{1}{y}) + (x + \frac{1}{x}) + l + 2} \ dx \ dy.\]

According to Poincaré residue theorem, \(\frac{d\tilde{m}(P_l)}{dl}\) is a period of \(P_l(x, y) = 0\). We know that periods of first species differentials on the compact Riemann surface associated with the curve \(P_l(x, y) = 0\) verify a linear differential equation of second order given by Gauss-Manin’s connexion. It is called Picard-Fuchs’ differential equation

\[\Omega'' + P\Omega' + Q\Omega = 0,\]

where \(P\) and \(Q\) are rational functions of \(l\).

We determine the equation verified by \(m'(l)\) by using Stiller’s method. So we get for the family, the differential equation

\[g'' + \frac{-48 + 16l + 3l^2}{l(l + 12)(l - 4)} g' + \frac{l^2 + 2l + 12}{l(l + 12)(l - 4)} g = 0.\]

By Frobenius [6] this differential equation admits, in a vicinity of infinity, a logarithmic solution and a solution with entire series expansion of the form \(Tf(T)\) with \(T = \frac{1}{l}\), and \(f(T)\) with entire series expansion in a vicinity of \(T = 0\). We check easily that the family of polynomials do not vanish on the torus for \(l > 4\). So \(m'(l) = \frac{2}{\pi \sqrt{l^2 + 12l}} K\left(\sqrt{\frac{16}{l+12}}\right)\) is a regular solution of the differential equation in a vicinity of \(+\infty\).

2. According to (1), by integration of \(m'(l)\) between \(l\) and \(s\) we have

\[\int_{l}^{s} m'(l) \ dl = \frac{8}{\pi} \int_{s'}^{k} \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} dk + 4 \int_{s'}^{k} \frac{1}{k\sqrt{4 - 3k^2}} dk \text{ avec } s' = \sqrt{\frac{16}{s+12}},\]

so

\[m(s) - m(l) = \frac{8}{\pi} \int_{s'}^{k} \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} dk + 2\alpha_t(s),\]

with

\[\alpha_t(s) = \log \sqrt{s} + \log \sqrt{\frac{1 + \frac{12}{s}}{4}} + \log \left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \sqrt{4 - \frac{48}{s + 12}}\right) - \log \left(\sqrt{l + 12} + \sqrt{l}\right) + \log(2\sqrt{3}).\]
When $s \to +\infty$, $m(s) \sim \log s$, then

$$m(l) = -\frac{8}{\pi} \int_0^k \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} dk - 2\log 2 + 2\log \left(\sqrt{l + 12} + \sqrt{l}\right).$$

3. According to (2), put $l = 4$, we have

$$m(4) = 2\log 3 - \frac{8}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} dk.$$

4. If $l < -12$ then $\Delta > 0$, so

$$m(l) = \frac{1}{\pi} \int_0^\pi \log \left| (4\cos s + l + 2) + \sqrt{\Delta} \right| ds.$$

then

$$m'(l) = -\frac{1}{\pi} \int_0^\pi \frac{ds}{\sqrt{l}(l + 8\cos s + 4)}.$$

Set $X = \cos \frac{s}{2}$, it comes

$$m'(l) = -\frac{2}{\pi} \frac{1}{\sqrt{l^2 - 4l}} \int_0^1 \frac{dX}{\sqrt{(1 - X^2)(1 - \frac{16}{4-l}X^2)}}.$$

Set $k^2 = \frac{16}{4-l}$, we have

$$m'(l) = -\frac{2}{\pi} \frac{1}{\sqrt{l^2 - 4l}} K\left(\sqrt{\frac{16}{4-l}}\right).$$

As $m'(l)$ admits an entire series expansion in a vicinity of $-\infty$, it is the regular solution of the Picard-Fuchs differential equation associated to the family.

5. According to (4), we have $k^2 = \frac{16}{4-l}$ then $dl = \frac{32}{k^3} dk$.

So

$$m'(l) dl = \frac{-8}{\pi} \frac{1}{k\sqrt{4 - k^2}} K dk.$$

By integration between $s$ and $l$, we have

$$\int_s^l m'(l) dl = \frac{-8}{\pi} \int_{s'}^k \frac{K}{k\sqrt{4 - k^2}} dk \text{ avec } s' = \frac{4}{\sqrt{4 - s}}.$$
then

\[ m(l) - m(s) = -\frac{8}{\pi} \int_{s'}^{s} \frac{K - \frac{\pi}{2}}{k \sqrt{4 - k^2}} dk - 4 \int_{s'}^{s} \frac{1}{k \sqrt{4 - k^2}} dk. \]

We prove that

\[ \int_{s'}^{s} \frac{1}{k \sqrt{4 - k^2}} dk = -\frac{1}{2} \left( \log \left( \frac{2}{k} + \sqrt{\frac{4}{k^2} - 1} \right) - \log \left( \frac{2}{s'} + \sqrt{\frac{4}{s'^2} - 1} \right) \right). \]

When \( s \to -\infty, s' \to 0 \) and \( m(s) \sim \log(-s) \), then

\[ m(l) = -\frac{8}{\pi} \int_{0}^{1} \frac{K(k^2) - \frac{\pi}{2}}{k \sqrt{4 - k^2}} dk + 2 \log \frac{\sqrt{4 - l} + \sqrt{-l}}{2}. \]

6. For \( l < -12 \), we have

\[ m(l) = -\frac{8}{\pi} \int_{0}^{1} \frac{K(k^2) - \frac{\pi}{2}}{k \sqrt{4 - k^2}} dk + 2 \log \frac{\sqrt{4 - l} + \sqrt{-l}}{2}. \]

So

\[ m(-12) = -\frac{8}{\pi} \int_{0}^{1} \frac{K - \frac{\pi}{2}}{k \sqrt{4 - k^2}} dk + 2 \log \left( 2 + \sqrt{3} \right). \]

7. If \( 0 < l < 4 \), put \( a = \arccos \left( -\frac{4 - l}{8} \right) \); then \( \Delta > 0 \) if \( 0 < s < a \) and \( \Delta \leq 0 \), if \( a \leq s < \pi \).

So

\[ m(l) = \frac{1}{\pi} \left[ \int_{0}^{a} \log \left| 4 \cos s + l + 2 + \sqrt{\Delta} \right| ds + \int_{a}^{\pi} \log \left| 2 \cos s + 1 \right| ds \right]. \]

By differentiating with respect to \( l \),

\[ m'(l) = \frac{1}{\pi \sqrt{l}} \int_{0}^{a} \frac{ds}{\sqrt{l + 8 \cos s + 4}}. \]

Put \( X = \frac{1}{\sin \frac{s}{2}} \), it comes

\[ m'(l) = \frac{-1}{2\pi \sqrt{l}} \int_{-\infty}^{\infty} \frac{dX}{\sqrt{l + 12 - \frac{l}{X^2 - 1}} \left( \frac{l + 12}{16} X^2 - 1 \right)}. \]
and if \( k^2 = \frac{l + 12}{16} \), then

\[
m'(l) = -\frac{1}{2\pi \sqrt{l}} \int_{\frac{1}{k}}^{+\infty} \frac{dX}{\sqrt{(1 - X^2)(1 - k^2 X^2)}}.
\]

So

\[
m'(l) = \frac{1}{2\pi \sqrt{l}} K\left(\sqrt{\frac{l + 12}{16}}\right).
\]

8. If \( 0 < l < 4 \), we have \( k^2 = \frac{l + 12}{16} \) then \( dl = 32kd\theta \), so

\[
m'(l) \, dl = \frac{1}{2\pi \sqrt{l}} 32kK \, dk.
\]

By integration between 0 and 4, we obtain

\[
\int_0^4 m'(l) \, dl = \frac{8}{\pi} \int_1^1 \frac{K}{\sqrt{4k^2 - 3}} k \, dk.
\]

Put \( u = \sqrt{4k^2 - 3} \), so

\[
m(4) - m(0) = \frac{2}{\pi} \int_0^1 K\left(\sqrt{\frac{u^2 + 3}{4}}\right) \, du.
\]

As \( m(0) = 0 \), then

\[
m(4) = \frac{2}{\pi} \int_0^1 K\left(\sqrt{\frac{u^2 + 3}{4}}\right) \, du
\]

9. If \( -12 < l < 0 \), put \( a = \arccos\left(-\frac{4 + l}{8}\right) \); then \( \Delta > 0 \) if \( a < s < \pi \) and \( \Delta \leq 0 \), if \( 0 < s \leq a \).

So

\[
m(l) = \frac{1}{\pi} \left[ \int_0^a \log|2\cos s + 1| \, ds + \int_a^\pi \log|-4\cos s - l - 2 + \sqrt{\Delta}| \, ds \right].
\]

By differentiating with respect to \( l \), we obtain

\[
m'(l) = \frac{-1}{\pi} \int_a^\pi \frac{ds}{\sqrt{l(l + 8\cos s + 4)}}.
\]
Put \( X = \frac{1}{\cos \frac{s}{2}} \), we have

\[
m'(l) = \frac{1}{2\pi \sqrt{-l}} \int_{\frac{1}{k}}^{+\infty} \frac{dX}{\sqrt{(1 - X^2) \left( 1 - \left( \frac{4 - l}{16} \right) X^2 \right)}},
\]

and if \( k^2 = \frac{4 - l}{16} \), it comes

\[
m'(l) = \frac{-1}{2\pi \sqrt{-l}} K \left( \sqrt{\frac{4 - l}{16}} \right).
\]

10. For \(-12 < l < 0\), we have \( k^2 = \frac{4 - l}{16} \) and \( dl = -32kdk \).

By integration between \(-12\) and 0, we obtain

\[
\int_{-12}^{0} m'(l) \, dl = -\frac{8}{\pi} \int_{\frac{1}{2}}^{1} \frac{K k}{\sqrt{4k^2 - 1}} \, dk.
\]

Put \( u = \sqrt{4k^2 - 1} \), we have

\[
\int_{-12}^{0} m'(l) \, dl = -\frac{2}{\pi} \int_{0}^{\sqrt{3}} K \left( \sqrt{\frac{u^2 + 1}{4}} \right) \, du.
\]

Put \( u = \sqrt{3t} \), it comes

\[
\int_{-12}^{0} m'(l) \, dl = -\frac{2\sqrt{3}}{\pi} \int_{0}^{1} K \left( \sqrt{\frac{3t^2 + 1}{4}} \right) \, dt.
\]

As \( m(0) = 0 \), then

\[
m(-12) = \frac{2\sqrt{3}}{\pi} \int_{0}^{1} K \left( \sqrt{\frac{3t^2 + 1}{4}} \right) \, dt.
\]

\[\square\]

**Corollary 3.2.**

\[
L'(\chi_{-4}, -1) = \frac{3}{4\pi} \int_{0}^{1} K \left( \sqrt{\frac{u^2 + 3}{4}} \right) \, du.
\]
\[ L'(\chi_4, -1) = \frac{3}{4} \log(3) - \frac{3}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} \, dk. \]

\[ L'(\chi_3, -1) = \frac{3\sqrt{3}}{10 \pi} \int_0^1 K\left(\frac{3t^2 + 1}{4}\right) \, dt. \]

\[ L'(\chi_3, -1) = -\frac{6}{5 \pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - k^2}} \, dk + \frac{3}{10} \log(2 + \sqrt{3}). \]

**Proof.** According [5], the explicit values \( m(4) \) and \( m(-12) \) are

\[ m(4) = \frac{8}{3} L'(\chi_4, -1) \]

and

\[ m(-12) = \frac{20}{3} L'(\chi_3, -1). \]

Then

\[ L'(\chi_4, -1) = \frac{3}{4 \pi} \int_0^1 K\left(\sqrt{\frac{u^2 + 3}{4}}\right) \, du. \]

\[ L'(\chi_4, -1) = \frac{3}{4} \log(3) - \frac{3}{\pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - 3k^2}} \, dk. \]

\[ L'(\chi_3, -1) = \frac{3\sqrt{3}}{10 \pi} \int_0^1 K\left(\frac{3t^2 + 1}{4}\right) \, dt. \]

\[ L'(\chi_3, -1) = -\frac{6}{5 \pi} \int_0^1 \frac{K - \frac{\pi}{2}}{k\sqrt{4 - k^2}} \, dk + \frac{3}{10} \log(2 + \sqrt{3}). \]

\[ \square \]
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