Isomorphism Theorems for Variants of Semigroups of Linear Transformations

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Abstract

If $S$ is a semigroup and $a \in S$, the semigroup $(S, \circ)$ defined by $x \circ y = xay$ for all $x, y \in S$ is called a variant of $S$ and $(S, \circ)$ is denoted by $(S, a)$. In 2003-2004, Tsyaputa characterized when two variants of the following transformation semigroups are isomorphic: the symmetric inverse semigroup, the full transformation semigroup and the partial transformation semigroup on a finite nonempty set. In this paper, we consider the semigroups under composition of all linear transformations of a finite-dimensional vector space over a finite field. We determine when its variants are isomorphic. We also obtain as a consequence in the same matter for the full $n \times n$ matrix semigroup over a finite field.

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1 Introduction and Preliminaries

The cardinality of a set $X$ is denoted by $|X|$. The value of the mapping $\alpha$ at $x$ in the domain of $\alpha$ shall be written as $x \alpha$. The range (image) of $\alpha$ is denoted by ran $\alpha$.

If $S$ is a semigroup and $a \in S$, the semigroup $(S, \circ)$ defined by $x \circ y = xay$ for all $x, y \in S$ is called a variant of $S$ and it is denoted by $(S, a)$. Variants of abstract semigroups were first studied by Hickey [2] in 1983. In fact, variants of concrete semigroups of relations were earlier considered by Magill [8] in 1967. Hickey [2, 1, 3, 4, 5] introduced various results relating to variants of semigroups. Khan and Lawson [7] determined an element $a$ in a regular
semigroup with identity and an inverse semigroup such that \((S, a)\) is a regular semigroup.

It is interesting to know when two variants of a certain semigroup are isomorphic. It is clear that if \(S\) is a semigroup with identity and \(a, b\) are units of \(S\), \((S, a) \cong (S, b)\) through the mapping \(x \mapsto axb^{-1}\), in particular, \((S, a) \cong S\) through \(x \mapsto ax\). In this case, \(a^{-1}\) is the identity of \((S, a)\). Moreover, if \(S\) has a zero 0, then 0 is clearly the zero of the variant \((S, a)\) of \(S\) for every \(a \in S\).

For a nonempty set \(X\), let \(T(X), P(X)\) and \(I(X)\) denote the full transformation semigroup, the partial transformation semigroup and the symmetric inverse semigroup. Notice that \(T(X), P(X)\) and \(I(X)\) are subsemigroups of \(P(X)\). If \(X\) is a finite set containing \(n\) elements, let \(T_n, P_n\) and \(I_n\) stand for \(T(X), P(X)\) and \(I(X)\), respectively. For \(\alpha \in P_n\) and \(k \in \{1, \ldots, n\}\), let

\[\alpha_k = |\{y \in \text{ran } \alpha \mid |y\alpha^{-1}| = k\}|.

The \(n\)-tuple \((\alpha_1, \ldots, \alpha_n)\) is called the type of \(\alpha\). In 2003-2004, Tsyaputa [9, 10] provided the remarkable results on variants of \(I_n, T_n\) and \(P_n\) as follows: for \(\alpha, \beta \in I_n, (I_n, \alpha) \cong (I_n, \beta)\) if and only if \(|\text{ran } \alpha| = |\text{ran } \beta|\), for \(\alpha, \beta \in T_n, (T_n, \alpha) \cong (T_n, \beta)\) if and only if \(\alpha\) and \(\beta\) have the same type and this is true for variants of \(P_n\).

We recall some notations and basic knowledge in linear algebra. Let \(V\) and \(W\) be vector spaces over a field \(F\). Let \(L_F(V, W)\) be the set of all linear transformations \(\alpha : V \rightarrow W\) and let \(L_F(V)\) stand for \(L_F(V, V)\). Then \(L_F(V)\) is a semigroup under composition. Note that \(1_V\), the identity mapping on \(V\) and \(0_V\), the zero mapping on \(V\) are the identity and the zero of the semigroup \(L_F(V)\), respectively. If \(\theta \in L_F(V, W)\), then \(\dim_F V = \dim_F \ker \theta + \dim_F \text{ran } \theta\). We call \(\dim_F \ker \theta\) and \(\dim_F \text{ran } \theta\) the nullity and the rank of \(\theta\), respectively and they are denoted respectively by \(\text{nullity } \theta\) and \(\text{rank } \theta\). If \(B\) is a basis of \(V\), \(B'\) is a basis of \(W\) and \(\theta \in L_F(V, W)\) is such that \(\theta|_B\) is a bijection from \(B\) onto \(B'\), then \(\theta\) is an isomorphism from \(V\) onto \(W\). If \(B_1\) is a basis of \(\ker \theta\) and \(B\) a basis of \(V\) containing \(B_1\), then \((B \setminus B_1)\theta\) is a basis of \(\text{ran } \theta\) and for distinct \(u, v \in B \setminus B_1, u \theta \neq v \theta\). In particular, if \(W\) is a subspace of \(V\), \(B_1\) a basis of \(W\) and \(B\) a basis of \(V\) containing \(B_1\), then \(\{v + W \mid v \in B \setminus B_1\}\) is a basis of the quotient space \(V/W\) and for distinct \(u, v \in B \setminus B_1, u + W \neq v + W\).

It is clear that for a basis \(B\) of \(V\),

\[|L_F(V, W)| = |\{\alpha \mid \alpha : B \rightarrow W\}| = |W|^{|B|}.

In particular, if \(V\) and \(W\) are finite-dimensional, \(F\) is finite, \(\dim_F V = n\) and \(\dim_F W = k\), then \(W \cong F^k\) as vector spaces and hence

\[|L_F(V, W)| = |F^k|^n = |F|^{(\dim_F V)(\dim_F W)} < \infty.\]
For a positive integer $n$ and a field $F$, let $M_n(F)$ be the multiplicative semigroup of all $n \times n$ matrices over a field $F$. If $V$ is finite-dimensional and $\dim_F V = n$, then there exists a semigroup isomorphism $\varphi : L_F(V) \rightarrow M_n(F)$ which preserves ranks ([6], p. 330 and 336-337).

In this paper, we shall prove that if $V$ is a finite-dimensional vector space over a finite field $F$ and $\theta_1, \theta_2 \in L_F(V)$, then $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ if and only if rank $\theta_1 = \text{rank} \theta_2$. As a consequence, we have that if $F$ is a finite field and $P_1, P_2 \in M_n(F)$, then $(M_n(F), P_1) \cong (M_n(F), P_2)$ if and only if rank $P_1 = \text{rank} P_2$.

\section{Main Result}

To prove the main result, the following series of lemmas is needed.

\textbf{Lemma 2.1.} Let $S$ be a semigroup with identity and $a, b \in S$. If there exist units $u, v$ in $S$ such that $uav = b$, then $(S, a) \cong (S, b)$.

\textit{Proof.} Define $\varphi : S \rightarrow S$ by $x\varphi = v^{-1}xu^{-1}$ for all $x \in S$. It is evident that $\varphi$ is a bijection. If $x, y \in S$, then

$$(xay)\varphi = v^{-1}xayu^{-1} = v^{-1}xu^{-1}uavv^{-1}yu^{-1} = v^{-1}xu^{-1}bv^{-1}yu^{-1} = (x\varphi)b(y\varphi).$$

Thus $\varphi$ is an isomorphism from $(S, a)$ onto $(S, b)$. \hfill \Box

\textbf{Lemma 2.2.} Let $V$ be a vector space over a field $F$ and $\theta_1, \theta_2 \in L_F(V)$. If rank $\theta_1 = \text{rank} \theta_2$, nullity $\theta_1 = \text{nullity} \theta_2$ and $\dim_F(V/\text{ran} \theta_1) = \dim_F(V/\text{ran} \theta_2)$, then there exist isomorphisms $\varphi, \psi \in L_F(V)$ such that $\varphi \theta_1 \psi = \theta_2$.

\textit{Proof.} Let $B_1$ and $B_2$ be bases of $\ker \theta_1$ and $\ker \theta_2$, respectively, and let $\tilde{B}_1$ be a basis of $V$ containing $B_1$ and $\tilde{B}_2$ a basis of $V$ containing $B_2$. It follows that $(\tilde{B}_1 \setminus B_1)\theta_1$ and $(\tilde{B}_2 \setminus B_2)\theta_2$ are bases of $\text{ran} \theta_1$ and $\text{ran} \theta_2$, respectively. We also have $|(\tilde{B}_1 \setminus B_1)\theta_1| = |\tilde{B}_1 \setminus B_1|$ and $|(\tilde{B}_2 \setminus B_2)\theta_2| = |\tilde{B}_2 \setminus B_2|$. Next let $\tilde{B}_1$ be a basis of $V$ containing $(\tilde{B}_1 \setminus B_1)\theta_1$ and $\tilde{B}_2$ a basis of $V$ containing $(\tilde{B}_2 \setminus B_2)\theta_2$. By assumption, $|\tilde{B}_1| = |\tilde{B}_2|$ and $|(\tilde{B}_1 \setminus B_1)\theta_1| = |(\tilde{B}_2 \setminus B_2)\theta_2|$. Therefore there exists an isomorphism $\varphi \in L_F(V)$ such that $\varphi \theta_1 = \tilde{B}_1 \setminus B_1$ and $\varphi \theta_2 = \tilde{B}_1 \setminus B_1$. Since $\dim_F(V/\text{ran} \theta_1) = \dim_F(V/\text{ran} \theta_2)$, it follows that $|\tilde{B}_1 \setminus (\tilde{B}_1 \setminus B_1)\theta_1| = \dim_F(V/\text{ran} \theta_1) = \dim_F(V/\text{ran} \theta_2) = |\tilde{B}_2 \setminus (\tilde{B}_2 \setminus B_2)\theta_2|$. Let $\pi : \tilde{B}_1 \setminus (\tilde{B}_1 \setminus B_1)\theta_1 \rightarrow \tilde{B}_2 \setminus (\tilde{B}_2 \setminus B_2)\theta_2$ be a bijection. Note that $\tilde{B}_1 = (\tilde{B}_1 \setminus B_1)\theta_1 \cup (\tilde{B}_1 \setminus (\tilde{B}_1 \setminus B_1)\theta_1) = ((\tilde{B}_2 \setminus B_2)\varphi \theta_1) \cup (\tilde{B}_1 \setminus (\tilde{B}_1 \setminus B_1)\theta_1)$. \hfill \Box
Define $\psi \in L_F(V)$ on $\tilde{B}_1$ by

$$
\psi = \begin{pmatrix}
    (u_\varphi)_{\theta_1} & v \\
    u_{\theta_2} & u_{\pi},
\end{pmatrix}
$$

Since $u_{\theta_2} \neq v_{\theta_2}$ for distinct $u, v \in \tilde{B}_2 \setminus B_2$, it follows that $\psi|_{\tilde{B}_1}$ is a bijection from $\tilde{B}_1$ onto $\tilde{B}_2$. Hence $\psi$ is an isomorphism of $V$. If $u \in B_2$, then $u_\varphi \in B_1$, so $u_\varphi \psi_1 \psi = 0 = u_{\theta_2}$. If $u \in B_2 \setminus B_2$, then by the definition of $\psi$, $u_\varphi \psi_1 \psi = u_{\theta_2}$. Therefore we have $\varphi \psi_1 \psi = \theta_2$, as desired.

**Lemma 2.3.** Let $V$ be a finite-dimensional vector space over a field $F$ and $\theta_1, \theta_2 \in L_F(V)$. If $\text{rank} \, \theta_1 = \text{rank} \, \theta_2$, then there exist isomorphisms $\varphi, \psi \in L_F(V)$ such that $\varphi \theta_1 \psi = \theta_2$.

*Proof.* Since $\text{dim} \, F \, V = \text{nullity} \, \theta_1 + \text{rank} \, \theta_1 = \text{nullity} \, \theta_2 + \text{rank} \, \theta_2$, $\text{dim} \, F \, V$ is finite and $\text{rank} \, \theta_1 = \text{rank} \, \theta_2$, it follows that $\text{nullity} \, \theta_1 = \text{nullity} \, \theta_2$. Also, we have $\text{dim} \, F \, V/\text{ran} \, \theta_1 = \text{dim} \, F \, V - \text{rank} \, \theta_1 = \text{dim} \, F \, V - \text{rank} \, \theta_2 = \text{dim} \, F \, V/\text{ran} \, \theta_2)$. Hence by Lemma 2.2, the desired result follows.

Notice that the converse of Lemma 2.3 is clearly true.

The following lemma follows directly from Lemma 2.1 and Lemma 2.3.

**Lemma 2.4.** Let $V$ be a finite-dimensional vector space over a field $F$ and $\theta_1, \theta_2 \in L_F(V)$. If $\text{rank} \, \theta_1 = \text{rank} \, \theta_2$, then $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$.

**Theorem 2.5.** Let $V$ be a finite-dimensional vector space over a finite field $F$ and $\theta_1, \theta_2 \in L_F(V)$. Then $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ if and only if $\text{rank} \, \theta_1 = \text{rank} \, \theta_2$.

*Proof.* First assume that $(L_F(V), \theta_1) \cong (L_F(V), \theta_2)$ through an isomorphism $\varphi$. Since $0_V$ is the zero of both $(L_F(V), \theta_1)$ and $(L_F(V), \theta_2)$, we have that $0_V \varphi = 0_V$. We claim that $\alpha \theta_1 = \beta \theta_1$ if and only if $(\alpha \varphi) \theta_2 = (\beta \varphi) \theta_2$ for all $\alpha, \beta \in L_F(V)$. Let $\alpha, \beta \in L_F(V)$ and assume that $\alpha \theta_1 = \beta \theta_1$. Then $\alpha \theta_1 \lambda = \beta \theta_1 \lambda$ for all $\lambda \in L_F(V)$, it follows that $(\alpha \varphi) \theta_2 (\lambda \varphi) = (\beta \varphi) \theta_2 (\lambda \varphi)$ for all $\lambda \in L_F(V)$. Since $(L_F(V)) \varphi = L_F(V)$, we have $(\alpha \varphi) \theta_2 = (\alpha \varphi) \theta_2 1_V = (\beta \varphi) \theta_2 1_V = (\beta \varphi) \theta_2$. But since $\varphi^{-1}$ is an isomorphism from $(L_F(V), \theta_2)$ onto $(L_F(V), \theta_1)$, if $(\alpha \varphi) \theta_2 = (\beta \varphi) \theta_2$, then from the above proof we have similarly that $(\alpha \varphi) \varphi^{-1} \theta_1 = (\beta \varphi) \varphi^{-1} \theta_1$, i.e., $\alpha \theta_1 = \beta \theta_1$. Therefore we prove that $\alpha \theta_1 = \beta \theta_1$ if and only if $(\alpha \varphi) \theta_2 = (\beta \varphi) \theta_2$. In particular, if $\beta = 0_V$, then $\alpha \theta_1 = 0_V$ if
and only if \((\alpha\varphi)\theta_2 = 0_V\). This proves that for every \(\alpha \in L_F(V)\), \(\alpha\theta_1 = 0_V\) if and only if \((\alpha\varphi)\theta_2 = 0_V\). It follows that \(\text{ran} \alpha \subseteq \text{ker} \theta_1\) if and only if \(\text{ran} \alpha \varphi \subseteq \text{ker} \theta_2\) for all \(\alpha \in L_F(V)\). This proves that \((L_F(V, \text{ker} \theta_1)) \varphi = L_F(V, \text{ker} \theta_2)\). Consequently, \(|L_F(V, \text{ker} \theta_1)| = |L_F(V, \text{ker} \theta_2)|\). As mentioned in Section 1, \(|L_F(V, \text{ker} \theta_1)| = |F|^{(\text{dim}_F V)(\text{nullity} \theta_1)}\) and \(|L_F(V, \text{ker} \theta_2)| = |F|^{(\text{dim}_F V)(\text{nullity} \theta_2)}\).

It follows that \(\text{nullity} \theta_1 = \text{nullity} \theta_2\). Hence \(\text{rank} \theta_1 = \text{dim}_F V - \text{nullity} \theta_1 = \text{dim}_F V - \text{nullity} \theta_2 = \text{rank} \theta_2\).

The converse follows directly from Lemma 2.4.

The proof is thereby completed. \(\square\)

**Corollary 2.6.** Let \(F\) be a finite field, \(n\) a positive integer and \(P_1, P_2 \in M_n(F)\). Then \((M_n(F), P_1) \cong (M_n(F), P_2)\) if and only if \(\text{rank} \ P_1 = \text{rank} \ P_2\).

**Proof.** Let \(V\) be a vector space over \(F\) of dimension \(n\). Then there exists a semigroup isomorphism \(\varphi : L_F(V) \rightarrow M_n(F)\) which preserves ranks. Let \(\theta_1, \theta_2 \in L_F(V)\) be such that \(\theta_1 \varphi = P_1\) and \(\theta_2 \varphi = P_2\). Then for all \(\alpha, \beta \in L_F(V)\),

\[
(\alpha \theta_1 \beta) \varphi = (\alpha \varphi) P_1 (\beta \varphi) \quad \text{and} \quad (\alpha \theta_2 \beta) \varphi = (\alpha \varphi) P_2 (\beta \varphi).
\]

Since \(\varphi : L_F(V) \rightarrow M_n(F)\) is a bijection, it follows from the above equalities that \(\varphi\) is an isomorphism from \((L_F(V), \theta_1)\) onto \((M_n(F), P_1)\) and an isomorphism from \((L_F(V), \theta_2)\) onto \((M_n(F), P_2)\), i.e., \((L_F(V), \theta_1) \cong (M_n(F), P_1)\) and \((L_F(V), \theta_2) \cong (M_n(F), P_2)\).

First assume that \((M_n(F), P_1) \cong (M_n(F), P_2)\). This implies that \((L_F(V), \theta_1) \cong (L_F(V), \theta_2)\). By Theorem 2.5, \(\text{rank} \ \theta_1 = \text{rank} \ \theta_2\). Since \(\varphi\) preserves ranks, it follows that \(\text{rank} \ P_1 = \text{rank} \ P_2\).

Conversely, assume that \(\text{rank} \ P_1 = \text{rank} \ P_2\). Then \(\text{rank} \ \theta_1 = \text{rank} \ \theta_2\) since \(\varphi\) preserves ranks. By Theorem 2.5, \((L_F(V), \theta_1) \cong (L_F(V), \theta_2)\). Consequently, \((M_n(F), P_1) \cong (M_n(F), P_2)\). \(\square\)

**Remark 2.7.** From Lemma 2.4 and the proof of Corollary 2.6 we can see that the following result holds: if \(F\) is a field (need not be finite), \(n\) a positive integer and \(P_1, P_2 \in M_n(F)\) are such that \(\text{rank} \ P_1 = \text{rank} \ P_2\), then \((M_n(F), P_1) \cong (M_n(F), P_2)\). In fact, the following result can be referred from Lemma 2.1 and the fact that if \(P_1, P_2 \in M_n(F)\) are such that \(\text{rank} \ P_1 = \text{rank} \ P_2\), then \(P_1\) is equivalent to \(P_2\), i.e., \(P_1 = Q_1 P_2 Q_2\) for some invertible matrices \(Q_1, Q_2\) in \(M_n(F)\) ([6], p. 338).

**References**


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