Automorphism Group of Groups

$Q_n$, $D_n$ and $L_{p,q}$

S. H. Jafari$^1$

Department of Mathematics
Shahrood University of Technology
P.O. Box 361995161-316, Shahrood, Iran
shjafari55@gmail.com

Abstract. Let $G$ be a finite $p$-group. A map $\varphi$ of $G$ to itself is an automorphism of $G$ if $\varphi$ is bijective and for any element $x, y \in G$, $\varphi(xy) = \varphi(x)\varphi(y)$. We find automorphisms groups of $Q_n$, $D_n$ and $L_{p,q}$. Also, we prove some properties of this group. We denote that $\text{Aut}(G)$ for these groups are meta-abelian and therefore are soluble of length 2.

Mathematics Subject Classification: 20D45, 20D15

1. Introduction

Let $G$ be a finite group. An automorphism $\sigma$ of $G$ is said to be central if and only if it induces identity on $G/Z(G)$, or equivalently, $g^{-1}\sigma(g) \in Z(G)$ for all $g \in G$, where $Z(G)$ is the center of $G$. The central automorphisms of $G$, denoted by $\text{Aut}_c(G)$, forms a normal subgroup of $\text{Aut}(G)$, the group of automorphisms of $G$. For a group $H$ and an abelian group $K$, $\text{Hom}(H, K)$ denotes the group of all homomorphisms from $H$ to $K$. For an element $g$ of $G$, the automorphism $i_g$ by definition $i_g(x) = gxg^{-1}$ for all $x \in G$, is said to be an inner automorphism. All inner automorphism of $G$, denoted by $\text{Inn}(G)$, forms a normal subgroup of $\text{Aut}(G)$. Notice that $\text{Aut}_c(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$.

The group of central automorphism group and inner automorphisms groups of finite groups are of great importance in the investigation of $\text{Aut}(G)$, and has been studied by several authors (see e.g., [1-6]).

$^1$The paper is supported by Shahrood University of Technology.
In this paper, we find $\text{Aut}(G)$, $\text{Aut_c}(G)$ and $\text{Inn}(G)$ for the groups $G = Q_n$, $D_n$ and $L_{p,q}$. Also we prove that $\text{Aut}(G)$ has a non-inner automorphism of order 2 when $G = D_n$ or $Q_n$ and $n$ is power of 2.

Throughout this paper all groups are assumed to be finite and the following notation is used: $\text{Hom}(G, H)$ denotes the group of homomorphism of $G$ into abelian group $H$; $Z_n$ the cyclic group of order $n$; $U_n$ the groups of units elements of $Z_n$. All unexplained notation is standard. Also, a non-abelian group that has no non-trivial abelian direct factor is said to be purely non-abelian.

2. Automorphism groups of $Q_n, D_n$

We begin this section to a theorem of presentation of groups that we shall use during this paper.

**Theorem 2.1.** Let $G$ be a group and $G = \langle X \mid R \rangle$ where $X$ is a set of generators and $R$ is a set of relations. Then a map $\varphi$ is an automorphism of $G$ if and only if $G = \langle \varphi(X) \rangle$ and $\varphi$ constant every relations of $R$ (i.e; if $x_1x_2...x_n = 1$ is an element of $R$ then $\varphi(x_1)\varphi(x_2)...\varphi(x_n) = 1$).

We give below some basic results that needed for our investigation of $\text{Aut}(G)$.

Consider the groups

$Q_n = \langle a, b \mid a^n = b^2, bab^{-1} = a^{-1} \rangle$

$D_n = \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle$.

then we have the following elementary lemma:

**Lemma 2.2.** We have

(a) $Q_n$ has a single element of order 2 and $|Z(Q_n)| = 2$,
(b) $Q_n$ and $D_n$ are nilpotent if and only if $n$ be a power of 2,
(c) $D_n$ has a non-trivial direct factor if and only if $n$ is even and $n/2$ is odd;

in this case $D_n \cong \mathbb{Z}_2 \times D_{n/2}$.

(d) Each element out of $\langle a \rangle$ in $Q_n$ is of order 4, and each element out of $\langle x \rangle$ in $D_n$

is of order 2.

**Lemma 2.3.** For $n > 2$

(a) $\langle a \rangle$ ch $Q_n$,
(b) $\langle x \rangle$ ch $D_n$.

**Proof.** Suppose that $H$ be a cyclic subgroup of order $2n$ and $H \neq \langle a \rangle$. Then we have $H\langle a \rangle = Q_n$ and so $|Q_n| = \frac{|H||\langle a \rangle|}{|H \cap \langle a \rangle|}$. Hence $|H \cap \langle a \rangle| = n$. But $H \cup \langle a \rangle$
$C_G(H \cap \langle a \rangle)$, then we have $H \cap \langle a \rangle \subseteq Z(Q_n)$, is a contradiction. (b) It is same to (a). \hfill \Box

**Corollary 2.4.** For $n > 2$, $|\text{Aut}(Q_n)| = 2n\phi(2n)$ and $|\text{Aut}(D_n)| = n\phi(n)$.

**Proof.** Let $\varphi \in \text{Aut}(Q_n)$. Then $o(\varphi(b)) = 4$, $\varphi(a) \in \langle a \rangle$ and $\varphi(b) \notin \langle a \rangle$. It follows that $|\text{Aut}(Q_n)| \leq 2n\phi(2n)$. Further more we notice that the map $\varphi$ by definition $a \mapsto a^i$, $b \mapsto ba^j$, where $(i, 2n) = 1$, is an automorphism of $Q_n$. \hfill \Box

**Theorem 2.5.** By definition $\varphi_i : x \mapsto x^i; y \mapsto y$ and $\psi_j : x \mapsto x; y \mapsto yx^j$, $H = \{\varphi_i | (i, 2n) = 1\}$ and $K = \{\psi_j | 0 \leq j \leq 2n - 1\}$,

we have

(a) $H \leq \text{Aut}(Q_n)$ and $K \leq \text{Aut}(Q_n)$,

(b) $K$ is cyclic and $\text{Aut}(Q_n) \cong K \rtimes H$.

**Proof.** (a) is elementary,

(b) Since $H \cap K = 1$ then $\text{Aut}(G) = HK$. In the other hand we observe that

$$\varphi_i^{-1}\psi_1\varphi_i(x) = \varphi_i^{-1}\psi_1(x^i) = \varphi_i^{-1}(x^i) = x,$$

$$\varphi_i^{-1}\psi_1\varphi_i(y) = \varphi_i^{-1}\psi_1(y) = \varphi_i^{-1}(xy) = \varphi_i^{-1}(x)\varphi_i^{-1}(y) = x^iy.$$ 

Therefore $H \leq \text{Aut}(Q_n)$, it is complete the proof. \hfill \Box

**Theorem 2.6.** By definition $\varphi_i : a \mapsto a^i; b \mapsto b$ and $\psi_j : a \mapsto a; b \mapsto ba^j$, $H = \{\varphi_i | (i, 2n) = 1\}$ and $K = \{\psi_j | 0 \leq j \leq n - 1\}$,

we have

(a) $H \leq \text{Aut}(D_n)$ and $K \leq \text{Aut}(D_n)$,

(b) $K$ is cyclic and $\text{Aut}(D_n)$ is semidirect of $K$ to $H$.

**Proof.** It is same to before theorem. \hfill \Box

**Theorem 2.7.** Suposse that $G$ is $D_n$ or $Q_n$, where $n > 2$. Then

(a) $\text{Aut}(G)$ is solvable of length 2.

(b) $\text{Aut}(G)$ is nilpotent if and only if $n$ be a power of 2.

**Proof.** It is obvies by theorem 2.5, 2.6.

(b) Let $n$ is not a power of 2. Hence $\text{Aut}(G)$ and then $\text{Aut}_c(G)$ has an element of order not 2. But $|\text{Aut}_c(G)| \leq |\text{Hom}(G/G'), Z(G))|$ and so $\text{Aut}_c(G)$ is elementary abelian 2-group, is a contradiction.

Converse, if $n$ be a power of 2 then $\text{Aut}(G)$ is power of 2 and hence is nilpotent. \hfill \Box

**Lemma 2.8.** If $n$ be power of 2 and $n > 2$ then $\text{Aut}_c(G) \cong Z_2 \times Z_2$
Corollary 2.9. If $n$ is power of $2$ and $n > 8$ then $G$ has at least one outer automorphism of order $2$.

Proof. We have $\text{Inn}(G) \cong G/Z(G) \cong D_k$ and so $Z(\text{Inn}(G)) \cong Z_2$. By lemma 2.2, $G$ is purely non-abelian group. Since $|\text{Aut}_c(G)| = |\text{Hom}(G/G', Z(G))| = 4$, then $\text{Aut}_c(G)$ is abelian and hence is isomorphic to $\text{Hom}(G/G', Z(G)) \cong Z_2 \times Z_2$. But $\text{Aut}_c(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$, it follows that $\text{Aut}_c(G)$ has an element of order $2$ out of $\text{Inn}(G)$.

3. Automorphism Groups of $L_{p,q}$

In this section we find $\text{Aut}(G), \text{Aut}(\text{Aut}(G)), ...$ for $G = L_{p,q}$. By definition

$$L_{p,q} = \langle x, y | x^p = y^q = 1, yxy^{-1} = x^i \rangle,$$

where $(p, q) = 1$, $q|p - 1$ and $t^q \equiv 1 (modp)$, we have

Lemma 3.1. By above definition

(a) $\langle x \rangle$ is characteristic subgroup of $L_{p,q}$,
(b) Any element out of $\langle x \rangle$ is of order $q$,
(c) $\text{Inn}(L_{p,q}) \cong L_{p,q}$ and $\text{Aut}_c(G) = 1$,
(d) If $\psi : x \mapsto x^i; y \mapsto y^kx^j$ is an automorphism of $L_{p,q}$ then $(i, p) = 1$ and $k = 1$.

Proof. (a),(b) and (c) are elementary. (d) Since $\psi(\langle x \rangle) = \langle x \rangle$ then $(i, p) = 1$. By Theorem 2.1 we should have $\psi(y)\psi(x)\psi(y)^{-1} = \psi(x)^t$ and then $x^{itk} = x^{it}$. Hence $it(t^{k-1}) \equiv 0 (modp)$. It follows that $q|t^{k-1} - 1$ and so $k = 1$.

Corollary 3.2. $|\text{Aut}(L_{p,q})| = p(p - 1)$.

Theorem 3.3. By assuming $\varphi_i : x \mapsto x^i; y \mapsto y$ and $\psi_j : x \mapsto x; y \mapsto yx^j$, $H = \{\varphi_i|1 \leq i \leq p - 1\}$ and $K = \{\psi_j|0 \leq j \leq p - 1\}$, we have

(a) $H \leq \text{Aut}(L_{p,q})$ and $K \leq \text{Aut}(L_{p,q})$,
(b) $K, H$ are cyclic and $\text{Aut}(L_{p,q})$ is semidirect of $K$ to $H$.

Proof. It is same to proofs of before section.

Corollary 3.4. $\text{Aut}(L_{p,q}) = \langle a, b | a^p = b^{p-1} = 1, bab^{-1} = a^s \rangle$ where $o(s) = p - 1$ in $U_p$.

Lemma 3.5. Let $G = \text{Aut}(L_{p,q})$. If $\psi : a \mapsto a^i; b \mapsto b^k a^j$ is an automorphism of $G$ then $k = 1$. 
Proof. By Theorem 2.1 we should have \( \psi(b)\psi(a)\psi(b)^{-1} = \psi(a)^s \) and then \( a^{is^k} = a^i \). Hence \( is(s^{k-1} - 1) \equiv 0(\text{mod} p) \). It follows that \( q|s^{k-1} - 1 \) and so \( k = 1 \).

**Theorem 3.6.** If \( G = \text{Aut}(L_{p,q}) \) then \( \text{Aut}(G) \cong G \).

Proof. By before lemma \( |\text{Aut}(G)| \leq p(p-1) \). Since \( \text{Aut}_c(L_{p,q}) = 1 \) then \( Z(G) = 1 \) and so \( \text{Inn}(G) \cong G \). It follows that \( |\text{Aut}(G)| = p(p-1) \) and then \( \text{Aut}(G) \cong G \).

**References**


Received: October, 2009