The Influence of Certain Permutable Subgroups on the Structure of Finite Groups

A. A. Heliel

Department of Mathematics, King Abdulaziz University
Faculty of Science 80203, Jeddah 21589, Saudi Arabia
heliel9@yahoo.com

S. M. Alharbia

Department of Mathematics, Umm Al-Qura University
Faculty of Science 8140, Makkah, Saudi Arabia

Abstract

A subgroup of a finite group $G$ is said to be $S$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$. A subgroup $H$ of a finite group $G$ is said to be $s$-semipermutable in $G$ if it permutes with every Sylow $p$-subgroup of $G$, where $p$ and the order of $H$ are relatively prime. A subgroup $H$ of a finite group $G$ is said to be $S$-quasinormally embedded in $G$ if every Sylow subgroup of $H$ is a Sylow subgroup of some $S$-quasinormal subgroup of $G$. In this paper, we are interested in studying the structure of the finite group $G$ under the assumption that certain subgroups of prime power order of $G$ are $s$-semipermutable or $S$-quasinormally embedded in $G$. Some recent results are improved and generalized.

Mathematics Subject Classification: 20D10; 20D20

Keywords: $s$-semipermutable subgroups, $S$-quasinormally embedded subgroups, $p$-nilpotent groups, $p$-solvable groups, $p$-supersolvable groups, supersolvable groups, saturated formations

---

1This research has been supported by the Institute of Scientific Research at Umm Al-Qura University, Makkah, Saudi Arabia.

2Permanent address: Department of Mathematics, Beni-Suef University, Faculty of Science 62511, Beni-Suef, Egypt
1 Introduction

Throughout only finite groups are considered. We use the standard notions and notations given by Doerk and Hawkes in [1].

We say, following Kegel [2], that a subgroup of a group $G$ is $S$-quasinormal in $G$ if it permutes with every Sylow subgroup of $G$. As a generalization of $S$-quasinormality, in [3], Chen introduced the following concept: A subgroup $H$ of a group $G$ is said to be $s$-semipermutable in $G$ if it permutes with every Sylow $p$-subgroup of $G$ with $(p, |H|) = 1$. Clearly, every $S$-quasinormal subgroup is $s$-semipermutable. The converse is not true in general and this can be easily seen by considering the alternating group of degree 4. Also, in [4], Ballester-Bolinches and Pedraza-Aguilera generalized the $S$-quasinormality as follows: A subgroup $H$ of a finite group $G$ is said to be $S$-quasinormally embedded in $G$ if every Sylow subgroup of $H$ is a Sylow subgroup of some $S$-quasinormal subgroup of $G$. It is easy for the reader to see that $S$-quasinormal subgroups are $S$-quasinormally embedded. The converse is not true in general; the symmetric group of degree 3 is a counterexample.

It seems that knowing some information about certain subgroups of prime power order of the group $G$ often helps us to understand the structure of $G$. For example, Buckley [5] proved that a group of odd order is supersolvable if all its minimal subgroups are normal (a subgroup of prime order is called a minimal subgroup). Srinivasan [6] proved that if the maximal subgroups of the Sylow subgroups of $G$ are normal in $G$, then $G$ is supersolvable. These important results on supersolvable groups have been generalized by many authors. One direction of the generalization is to replace the normality condition of minimal subgroups or maximal subgroups by a weaker condition; and the other direction is to minimize the number of minimal or maximal subgroups of Sylow subgroups of a finite group. For example, Shaalan [7] proved that if $G$ is a group and the cyclic subgroups of prime order or order 4 of $G$ are $S$-quasinormal in $G$, then $G$ is supersolvable. In [4], the authors studied and analyzed the influence of $S$-quasinormal embedded subgroups in the supersolvability of finite groups. Working within the framework of formation theory, Asaad and Heliel [8] generalized all the results in [4] (the reader is referred to [1] for notation and basic results in the theory of formation). For more results by using $S$-quasinormally embedded condition; see for instance [9] and [10]. In [11], L. Wang and Y. Wang studied the structure of the finite group $G$ under the assumption that the maximal and the minimal subgroups of the Sylow $p$-subgroups of $G$ are $s$-semipermutable. For more results by using $s$-semipermutable condition; see for instance [12], [13], [14] and [15].
The main object of this paper is to continue these investigations and determine the structure of the finite group $G$ under the assumption that certain subgroups of prime power order are $s$-semipermutable or $S$-quasinormally embedded in $G$.

## 2 Preliminaries

In this section, we collect some of the results that will be used later.

**Lemma 2.1**[3]. Let $G$ be a group:

(i) If $H$ is $s$-semipermutable subgroup in $G$ and $K$ is a subgroup of $G$ such that $H \leq K \leq G$, then $H$ is $s$-semipermutable in $K$.

(ii) Let $\pi$ be a set of primes, $N$ is a normal $\pi$-subgroup of $G$ and $H$ is $\pi$-subgroup of $G$. If $H$ is $s$-semipermutable in $G$, then $HN/N$ is $s$-semipermutable in $G/N$.

**Lemma 2.2**[15]. Let $G$ be a group and $H$ be a subgroup of $G$ contained in $O_p(G)$. Then $H$ is $S$-quasinormal in $G$ if and only if $H$ is $s$-semipermutable in $G$.

If $P$ is a finite $p$-group, we denote

$$\Omega(P) = \Omega_1(P) \text{ if } p > 2 \quad \text{and} \quad \Omega(P) = \langle \Omega_1(P), \Omega_2(P) \rangle \text{ if } p = 2,$$

where

$$\Omega_i(P) = \langle x \in P : O(x) = p^i \rangle.$$

**Lemma 2.3**[16]. Let $P$ be a Sylow 2-subgroup of the finite group $G$. If $P$ is quaternion-free and $\Omega_1(P) \leq Z(G)$, then $G$ is 2-nilpotent.

**Lemma 2.4**[15]. Let $G$ be a group and let $K$ be a normal subgroup of $G$ such that $G/K$ is supersolvable. If all subgroups of $K$ of prime order or order 4 are $s$-semipermutable in $G$, then $G$ is supersolvable.

**Lemma 2.5**[8]. Let $G$ be a group with a normal subgroup $H$ such that the maximal subgroups of the Sylow subgroups of $H$ are $S$-quasinormally embedded in $G$. Then for any nontrivial normal subgroup $N$ of $G$, the maximal subgroups of the Sylow subgroups of $HN/N$ are $S$-quasinormally embedded in $G/N$.

**Lemma 2.6**[8]. If $H$ is a core-free $S$-quasinormal subgroup of a group $G$ and $P$ is a Sylow $p$-subgroup of $H$, for some prime $p$, then $P$ is $S$-quasinormal in $G$.

**Lemma 2.7**[8]. Let $P$ be an elementary abelian $p$-subgroup of a group $G$ such that $P$ is not cyclic. Equivalent are:
(i) The subgroups of order $p$ in $P$ are normal in $G$.

(ii) The maximal subgroups of $P$ are normal in $G$.

Lemma 2.8[4]. Let $G$ be a group. If $U$ is $S$-quasinormally embedded subgroup in $G$ and $H$ is a subgroup of $G$ such that $U \leq H \leq G$, then $U$ is $S$-quasinormally embedded in $H$.

Lemma 2.9[8]. Let $G$ be a group and let $p$ be the smallest prime dividing the order of $G$. If the maximal subgroups of the Sylow $p$-subgroups of $G$ are $S$-quasinormally embedded in $G$, then $G$ is $p$-nilpotent.

For the saturated formation $\mathfrak{F}$, the $\mathfrak{F}$-hypercenter of a group $G$ will be denoted by $Z_{\mathfrak{F}}(G)$.

Lemma 2.10[17]. Let $H$ be a normal subgroup of a group $G$ such that $G/H \in \mathfrak{F}$, where $\mathfrak{F}$ is a saturated formation. If $\Omega(P) \leq Z_{\mathfrak{F}}(G)$, where $P$ is a Sylow $p$-subgroup of $H$, then $G/O_p(H) \in \mathfrak{F}$.

3 Main Results

The following result is a slight improvement of Theorem 3.2 in [15]:

Theorem 3.1. Let $G$ be a group and let $H$ be a normal subgroup of $G$ such that $G/H$ is $p$-nilpotent, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1)=1$. If there exists a Sylow $p$-subgroup $P$ of $H$ such that every cyclic subgroup of order $p$ or 4 (if $p=2$) is $s$-semipermutable in $G$, then $G$ is $p$-nilpotent.

Proof. Let $G$ be a counterexample of minimal order and let $M$ be a proper subgroup of $G$. Then $M/(M \cap H) \cong MH/H \leq G/H$, hence $M/(M \cap H)$ is $p$-nilpotent. Clearly, $M \cap H$ is a Sylow $p$-subgroup of $M \cap H$ and every cyclic subgroup of $M \cap H$ with order $p$ or 4 (if $p=2$) is $s$-semipermutable in $M$ by hypothesis and Lemma 2.1. Our choice of $G$ implies that $M$ is $p$-nilpotent. Thus $G$ is a minimal non $p$-nilpotent group (that is $G$ is not $p$-nilpotent and all its proper subgroups are $p$-nilpotent). By a result of Itô [18; IV, Satz 5.4], $G$ has a normal Sylow $p$-subgroup $G_p$ and a cyclic Sylow $q$-subgroup $G_q$ such that $G = G_p \rtimes G_q$. Moreover, $G_p$ is of exponent $p$ if $p > 2$ and of exponent at most 4 if $p = 2$. Since $G/H$ is $p$-nilpotent, $HG_q < G$. If $HG_q < G$, then $HG_q$ is nilpotent and so $G_q < HG_q$. Since $G_q$ char $HG_q$ and $HG_q < G$, it follows easily that $G_q < G$; a contradiction. Hence $HG_q = G$ and $P = G_p$. Now let $x$ be an element of $P = G_p$ with order $p$ or 4 (if $p = 2$). By hypothesis, $<x>$ is $s$-semipermutable in $G$. Since $P = G_p = O_p(G)$, it follows, by Lemma 2.2, that $<x>$ is $S$-quasinormal in $G$. Hence, $<x>G_q$ is a subgroup of $G$. However, $<x> = P \cap <x>G_q < <x>G_q$, hence $<x>G_q = <x> \times G_q$. Since $P = \Omega_1(P)$ if $P$
is of exponent $p$ and $P = \Omega_2(P)$ if $p$ is of exponent 4, it follows that $G = P \times G_q$; a contradiction. This completes the proof of the theorem.

Immediate consequence of Theorem 3.1, we have:

**Corollary 3.2.** Let $p$ be the smallest prime dividing the order of the group $G$ and let $P$ be a Sylow $p$-subgroup of $G$. If all subgroups of $P$ of order $p$ or order 4 are $s$-semipermutable in $G$, then $G$ is $p$-nilpotent.

For a group $G$, we define $D(G) = \cap\{H : H \triangleleft G$ and $G/H$ is nilpotent$\}$ and call it the nilpotent residual of $G$.

Now we can prove:

**Theorem 3.3.** Let $p$ be the smallest prime dividing the order of the group $G$ and let $P$ be a Sylow $p$-subgroup of $G$. If $P$ is quaternion-free and all minimal subgroups of $D(G) \cap P$ of order $p$ or order 4 are $s$-semipermutable in $G$, then $G$ is $p$-nilpotent.

**Proof.** Suppose that the result is false and let $G$ be a counterexample of minimal order. Then $G$ is not $p$-nilpotent and so $G$ contains a minimal non-$p$-nilpotent subgroup, $K$, say. By [18; IV, Satz 5.4], $K$ is a minimal nonnilpotent subgroup of $G$. By [18; IV, Satz 5.2], $|K| = p^nq^m$ for a prime $q \neq p$, $K$ has a normal Sylow $p$-subgroup $K_p$ of exponent $p$ when $p > 2$ or at most 4 if $p = 2$ and a non-normal cyclic Sylow $q$-subgroup $K_q$. Without loss of generality, we can assume that $K_p \leq P$. Clearly, $D(K) = K_p$. Then $D(K) \cap K_p = K_p$. By hypothesis, all minimal subgroups of $K_p$ are $s$-semipermutable in $G$. Then, by Lemma 2.1, all minimal subgroups of $K_p$ are $s$-semipermutable in $K$. In fact if the exponent of $K_p$ is $p$, then, by Corollary 3.2, $K$ is $p$-nilpotent; a contradiction. Thus, the exponent of $P$ is 4 and hence $p = 2$. So, by [18; III, Satz 5.2], $K_2' = Z(K_2) = \Phi(K_2)$, $K_2'$ is elementary abelian and $K_2/K_2'$ is a chief factor of $K$. Then $\Omega_1(K_2) = K_2' \leq Z(K_2)$. Now, by applying Lemma 2.3, we conclude that $K$ is 2-nilpotent; a final contradiction.

The following example shows that the hypothesis “$P$ is quaternion-free” is necessary in Theorem 3.3:

**Example 3.4.** Set $G = SL(2, 3)$. Then the Sylow 2-subgroup $P$ of $G$ is the quaternion group of order 8 and $D(G) = P$. Thus all minimal subgroups of $D(G) \cap P$ are normal ($s$-semipermutable) in $G$. However $G$ is not 2-nilpotent.

Immediate consequence of Theorem 3.3, we have:

**Corollary 3.5.** Let $p$ be the smallest prime dividing the order of $G$ and let $P$ be a Sylow $p$-subgroup of $G$. If all subgroups of $D(G) \cap P$ of order $p$ or order 4 are $s$-semipermutable in $G$, then $G$ is $p$-nilpotent.

Now we prove:
Theorem 3.6. If all minimal subgroups of \( D(G) \cap P \) are s-semipermutable in \( G \) for all Sylow subgroups \( P \) of \( G \), then \( G \) is supersolvable or \( G \) has a section isomorphic to the quaternion group of order 8.

Proof. If \( G \) has a section isomorphic to the quaternion group of order 8, then we are done. Thus we can assume that \( G \) has no section isomorphic to the quaternion group of order 8. Theorem 3.3 implies that \( G \) is \( r \)-nilpotent, where \( r \) is the smallest prime dividing the order of \( G \). Then \( G = RK \), where \( R \) is a Sylow \( r \)-subgroup of \( G \) and \( K \) is a normal Hall \( r \)-subgroup of \( G \). Clearly, \( D(K) \leq D(G) \). Since all minimal subgroups of \( D(K) \cap P \) are s-semipermutable in \( G \) by hypothesis, we have that all minimal subgroups of \( D(K) \cap P \) are s-semipermutable in \( K \) for all Sylow subgroups \( P \) of \( K \) by hypothesis. By induction on the order of \( G \), \( K \) is supersolvable. Hence \( Q \) is a normal Sylow \( q \)-subgroup of \( K \), where \( q \) is the largest prime dividing the order of \( K \). Since \( Q \) char \( K \triangleleft G \), we have that \( Q \triangleleft G \). Put \( D(G/Q) = L/Q \). Since \((G/Q)/(L/Q) \cong G/L \) is nilpotent, we have that \( D(G) \leq L \) and since \((G/Q)/(D(G)Q/Q) \cong G/(D(G))/(D(G)Q/Q) \) is nilpotent, we have that \( L \leq D(G)Q \). Hence \( L = D(G)Q \) and so \( D(G/Q) = D(G)Q/Q \). Clearly, \( D(G/Q) \cap (PQ/Q) = (D(G) \cap P)Q/Q \) for all Sylow subgroups \( PQ/Q \) of \( G/Q \). By hypothesis and Lemma 2.1, all minimal subgroups of \( D(G/Q) \cap (PQ/Q) \) are s-semipermutable in \( G/Q \) for all Sylow subgroups \( PQ/Q \) of \( G/Q \). By induction on the order of \( G \), \( G/Q \) is supersolvable. Since \( G/D(G) \) is nilpotent, we have that \( G/(D(G) \cap Q) \) is supersolvable. Applying Lemma 2.4, \( G \) is supersolvable. This completes the proof of the theorem.

The arguments which established Theorem 3.6 can be easily adapted to yield the following corollaries:

Corollary 3.7. If all subgroups of \( D(G) \cap P \) of prime order or order 4 are s-semipermutable in \( G \) for all Sylow subgroups \( P \) of \( G \), then \( G \) is supersolvable.

Corollary 3.8[7]. If all subgroups of \( P \) of prime order or order 4 are S-quasinormal in \( G \) for all Sylow subgroups \( P \) of \( G \), then \( G \) is supersolvable.

Corollary 3.9[5]. Assume that \( G \) is a group of odd order and every subgroup of \( G \) of prime order is normal in \( G \), then \( G \) is supersolvable.

Let \( p \) be a prime number. \( \mathcal{U}_p \) will denote to the class of all groups \( G \) such that every \( p \)-chief factor of \( G \) is cyclic (\( G \) is \( p \)-supersolvable), and \( \mathcal{U} \) will denote to the class of all supersolvable groups. Clearly, \( \mathcal{U}_p \) and \( \mathcal{U} \) are saturated formations. In the following, by using the concept of \( S \)-quasinormally embedded, we study the behaviour of \( \mathcal{U}_p \) as a class of \( p \)-supersolvable groups and apply the result to get information about the structure of the group through the theory of formation.
Theorem 3.10. Let $p$ be a prime, $G$ be a $p$-solvable group and let $H$ be a normal subgroup of $G$ such that $G/H \in \mathcal{U}_p$. If the maximal subgroups of the Sylow $p$-subgroups of $H$ are $S$-quasinormally embedded in $G$, then $G \in \mathcal{U}_p$.

Proof. Assume that the result is false and let $G$ be a counterexample of minimal order. By Lemma 2.5, we observe that the hypotheses of the Theorem are inherited by quotient groups. Since $\mathcal{U}_p$ is a saturated formation, the minimal choice of $G$ implies that $G$ is a monolithic primitive group such that $G/N \in \mathcal{U}_p$ and $\Phi(G) = 1$, where $N$ is the unique minimal normal subgroup of $G$.

Clearly, we may and shall assume that $N \leq H$. Since $G$ is $p$-solvable, it follows that either $N$ is a $p$-group or $p$-group. In the second case we have that $G \in \mathcal{U}_p$; a contradiction. Thus $N$ is a $p$-group (in particular elementary abelian $p$-group and $p^2$ divides $N$). Let $M$ be a complement of $N$ in $G$ and let $H_p$ be a Sylow $p$-subgroup of $H$. Then $H_p \cap M$ is a complement to $N$ in $H_p$. Pick a maximal subgroup $P_1$ of $H_p$ containing $H_p \cap M$. By hypotheses, $P_1$ is $S$-quasinormally embedded in $G$. Then there exists a subgroup $K$ which is $S$-quasinormal in $G$ and $P_1$ is a Sylow $p$-subgroup of $K$. If $N \leq K$, then $H_p = K(H_p \cap M) = K$ and this leads to $H_p = P_1$; a contradiction. Thus $N \not\leq K$. Therefore $K$ is core-free in $G$. By Lemma 2.6, $P_1$ is $S$-quasinormal in $G$ and this means that the Fitting subgroup $F(G)$ is nontrivial. Therefore $N \leq F(G)$ and since $G$ is a monolithic primitive group, it follows that $C_{G}(N) = N$. Also, $\Phi(F(G)) \leq \Phi(G) = 1$, so $F(G)$ is abelian. Hence $P_1 \leq F(G) = N$ and we have that $H_p = N$.

Let $P$ be any maximal subgroup of $H_p = N$. Using the hypotheses and the same argument above, it is easy to see that $P$ is $S$-quasinormal in $G$. Let $G_q$ be any Sylow $q$-subgroup of $G$ such that $(|G_q|, |N|) = 1$ and consider the subgroup $NG_q$. Since $P$ is $S$-quasinormal in $G$, it follows that $P$ is $S$-quasinormal in $NG_q$ and so $P \lhd NG_q$. Since $N$ is elementary abelian, it follows that the subgroups of order $p$ in $N$ are normal in $NG_q$ by Lemma 2.7. Since $N$ is normal in $G$, it follows that $N \cap Z(G_p) \neq 1$. Pick a subgroup $L$ of $N \cap Z(G_p)$ of order $p$. Then $L$ is normal in $G_p$ and, since $L$ is normal in $NG_q$ for any Sylow subgroup $G_q$ of $G$ with $(|G_q|, p) = 1$, it follows that $L$ is normal in $G$. Then $L = N$ as $N$ is a minimal normal subgroup of $G$. Thus $N$ is a self-centralizing cyclic minimal normal subgroup of $G$ and this implies that $G \in \mathcal{U}_p$; a final contradiction, completing the proof of the theorem.

The following example shows that Theorem 3.10 is not true if we omit the $p$-solvability of $G$.

Example 3.11. Consider $G = S_5$, the symmetric group of degree five, and $H = A_5$, the alternating group of degree five. Clearly, $G/H$ is 5-supersolvable and 1 is the maximal subgroup of any Sylow 5-subgroup of $H$ and it is certainly
normal (S-quasinormally embedded) in $G$. But $G$ does not 5-supersolvable.

**Remark 3.12.** A $\pi$-solvable group is $\pi$-supersolvable if its $\pi$-chief factors are all cyclic, i.e. if it is $p$-supersolvable for all primes $p \in \pi$. Clearly, results for $\pi$-supersolvability can be obtained just by taking the intersection of the corresponding results for $\pi$-supersolvability for all primes $p \in \pi$. One might ask whether Theorem 3.10 can be generalized by changing $p$ by $\pi$ to obtain a similar theorem to it in $\pi$-supersolvability, where $\pi$ is a set of prime numbers with $|\pi| > 1$. The answer is negative by the following example:

**Example 3.13.** Take $\pi = \{2, 3\}$ and consider the solvable group $G = S_4$, the symmetric group of degree 4, and $H = A_4$, the alternating group of degree 4. Obviously, $G/H$ is $\pi$-supersolvable and the maximal subgroups of the Hall $\pi$-subgroups of $H = O_\pi(H)$ are the Sylow subgroups of $H$ and they are $S$-quasinormally embedded in $G$. But $G$ is not $\pi$-supersolvable.

Immediate consequence of Theorem 3.10, we have:

**Corollary 3.14.** Let $p$ be a prime, $G$ be a $p$-solvable group and let $H$ be a normal subgroup of $G$ such that $G/H \in \mathcal{U}_p$. If the maximal subgroups of the Sylow $p$-subgroups of $H$ are $S$-quasinormal in $G$, then $G \in \mathcal{U}_p$.

Theorem 3.10 allows us to give new and short proof of Theorem 3.3 in [8] as follows:

**Theorem 3.15.** Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. Then $G \in \mathcal{F}$ iff there is a normal subgroup $H$ in $G$ such that $G/H \in \mathcal{F}$ and the maximal subgroups of the Sylow subgroups of $H$ are $S$-quasinormally embedded in $G$.

**Proof.** We need only to prove the “if” part. By Lemma 2.8, we have that the maximal subgroups of the Sylow subgroups of $H$ are $S$-quasinormally embedded in $H$. Repeated application of Lemma 2.9 implies that $H$ has a Sylow tower of supersolvable type. Let $p$ be the largest prime dividing the order of $H$ and let $H_p$ be a Sylow $p$-subgroup of $H$. It is clear that $H_p \trianglelefteq G$. Let $N$ be a minimal normal subgroup of $G$. If we assume that $G \notin \mathcal{F}$ and consider a counterexample of minimal order $G$, we can argue as in the proof of Theorem 3.10 to conclude that $G/N \in \mathcal{F}$ and $N$ is cyclic of order $p$. Then $N \leq Z_u(G)$ and, since $\mathcal{U} \subseteq \mathcal{F}$, it follows that $Z_u(G) \subseteq Z_\pi(G)$, so $N \leq Z_\pi(G)$. Hence $G \in \mathcal{F}$ by Lemma 2.10; a contradiction.

**References**


Received: July, 2010