On v-Hereditary Rings

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Abstract

In this paper, we introduce the notion of ”v-hereditary rings” which is a generalization of the notion of hereditary rings. Then we establish the transfer of this notion to trivial ring extensions and direct products and provide a class of v-hereditary rings which are not hereditary rings.

Mathematics Subject Classification: 16S50

Keywords: v-hereditary rings, hereditary rings, trivial ring extensions, direct products

1 Introduction

All rings considered below are commutative with unit and all modules are unital. Let $R$ be a commutative ring and let $Tot(R)$ denote the total ring of quotients of $R$. A ring $R$ is called a total ring of quotients if $R = Tot(R)$, that is every element of $R$ is invertible or zero-divisor. Let $I$ and $J$ be two nonzero ideals of $R$. We define the fractional ideal $(I : J) = \{ x \in Tot(R)/xJ \subset I \}$. We denote $(R : I)$ by $I^{-1}$. An ideal $I$ is said to be $v$-projectif if $I^{-1} = J^{-1}$ for some projectif ideal $J$ of $A$.

Recall that a ring $R$ is called hereditary if every ideal $I$ of $R$ is projectif. We introduce a new concept of a “v-hereditary” ring. A ring $R$ is called v-hereditary if every ideal of $R$ is $v$-projectif. An hereditary ring is naturally a v-hereditary ring.

Let $A$ be a ring and $E$ an $A$-module. The trivial ring extension of $A$ by $E$ is the ring $R := A \times E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Considerable work, part of it summarized in Glaz’s book [2] and Huckaba’s book (where $R$ is called the idealization of $E$ in $A$) [4], has been concerned with trivial ring extensions. See for instance
Our aim in this paper is to prove that v-hereditary rings are not hereditary rings, in general. Further, we investigate the possible transfer of the v-hereditary property to various trivial extension constructions and to direct products.

2 Main Results

The goal of this paper is to provide a classe of non-hereditary v-hereditary rings. But first, we give a wide class of v-hereditary rings.

**Theorem 2.1** Any total ring of quotients is v-hereditary.

**Proof.** Let \( R \) be a total ring. Our aim is to show that every ideal of \( R \) is v-projectif.

Let \( I \) be an ideal of \( R \). Then, \( I^{-1} = \{ x \in R/xI \subseteq R \} = R \) since \( R \) is a total ring. Hence, \( I^{-1} = R^{-1} \) and so \( I \) is v-projectif, as desired.

Examples of non-hereditary v-hereditary rings may stem from Theorem 2.1 as shown by the following two constructions.

**Example 2.2** Let \((A, M)\) be a local ring and \(E\) an \(A\)-module with \(ME = 0\). Let \(R := A \times E\) be the trivial ring extension of \(A\) by \(E\). Then:

1) \(R\) is a v-hereditary ring.

2) \(R\) is not an hereditary ring.

**Proof.** 1) By Theorem 2.1, it suffices to show that \(R\) is a total ring of quotients. Let \((a, e)\) be an element of \(R\). Two cases are then possibles:

If \(a \in A - M\) (that is \(a\) is invertible in \(A\)), then \((a, e)\) is invertible in \(R\) by [4, Theorem 25.1].

Now, assume that \(a \in M\). Then \((a, e)(0, f) = (0, 0)\) for all \(f \in E\) and so \((a, e)\) is a zero-divisor, as desired.

2) We claim that \(R\) is not hereditary. Deny. The ideal \(J := R(0, e)\) is projectif, where \(e(\neq 0) \in E\) and so \(J\) is free since \(R\) is local (by [4, Theorem 25.1] as \(A\) is local). A contradiction since \(J(0, f) = (0, 0)\) for all \(f \in E\). Hence, \(R\) is not hereditary, as desired.
Example 2.3 Let $Z$ be the ring of integers, $A := Z_2 \times Z$ be a countable direct sum of copies of $A/2A$ with addition and multiplication defined component wise, where $Z$ is the ring of integers. Let $R = A \times E$ with multiplication defined by $(a,e)(b,f) = (ab, af + be + ef)$. Then:
1) $R$ is a $v$-hereditary ring.
2) $R$ is not an hereditary ring.

Proof. 1) By Theorem 2.1, it suffices to show that $R$ is a total ring of quotients. Remark that $R$ has $(1, 0)$ as unit element. Let $(a,e)$ be an element of $R$. Without loss of generality, we may assume that $a = 1$ or $a \in 2Z$. Two cases are then possible:
Case 1: $a = 1$. Two cases are then possible:
If $e = 0$, then $(a,e)(0, 0) = (0, e + e) = (0, 2e) = (0, 0)$ since we have the four basic facts: $E$ is Boolean; $2E = 0$; $ae = e$ for any $a \in Z - 2Z$ and $e \in E$; and for any $e \neq 0 \in E$, there exists $f \neq 0 \in E$ such that $ef = 0$. Hence, $(1, e)$ is a zero-divisor element.
Case 2: $a \in 2Z$.
Let $f \in E$ such that $ef = 0$. Then $(a,e)(0, f) = (0, af + ef) = (0, 0)$ since $af = 0$ (as $a \in 2Z$) and $ef = 0$. Hence, $(a,e)$ is a zero-divisor element and this completes the proof of Example 2.3.

Next, we study the transfer of the $v$-hereditary property to direct products.

Proposition 2.4 Let $(R_i)_{i=1,\ldots,n}$ be a family of rings. Then, $\prod_{i=1}^{n} R_i$ is $v$-hereditary if and only if $R_i$ is $v$-hereditary for each $i = 1, \ldots, n$.

Proof. By induction on $n$, it suffices to prove the assertion for $n = 2$. It is clear to show that for any ideal $I_1 \prod I_2$ of $R_1 \prod R_2$, $(I_1 \prod I_2)^{-1} = (I_1)^{-1} \prod (I_2)^{-1}$.
Then, the conclusion follows easily from [5, Lemma 2.5] and this completes the proof of Proposition 2.4.

Now, we are able to construct non-local non-hereditary $v$-hereditary rings.

Example 2.5 Let $R_1$, $R_2$ be non-hereditary $v$-hereditary rings (see for instance Examples 2.2 and 2.3) and set $R = R_1 \prod R_2$. Then:
1) $R$ is a $v$-hereditary ring by Proposition 2.4.
2) It is clear that $R$ is non-local non-hereditary ring.

Finally, we construct non $v$-hereditary rings.
Example 2.6 Let $D$ be a local domain, $K := qf(D)$, $E$ be a $K$-vector space, and $R := D \times E$. Then $R$ is never an hereditary ring.

Proof. Observe that $(a, e) \in R$ is regular if and only if $a \neq 0$ (which means that $R - Z(R) = \{(a, e) \in R \mid a \neq 0\}$), then $T(R) = K \times E$.

Let $E'$ be a $D$-submodule of $E$ and set $J := 0 \times E'$ which is an ideal of $R$. To complete the proof, it suffices to show that there exists no projective ideal $H$ of $R$ such that $J^{-1} = H^{-1}$.

Let $H$ be an ideal of $R$ and set $I := \{a \in D/(a, e) \in H \text{ for some } e \in E\}$. Two cases are then possibles:

Case 1. $I = 0$. Then $H^{-1} = \{(a, e) \in T(R) = K \times E/(a, e)H \subseteq R\} = T(R)$ since $I = 0$. On the other hand, we claim that $H$ is not projective. Deny. Then $H$ is free since $R$ is local (by [4, Theorem 25.1] and since $D$ is local) and since $H$ is projective, a contradiction as $H(0, e) = 0$ for each $e \in E$. Hence, $H$ is not projective and $H^{-1} = T(R)$.

In particular, $J$ is not projective and $J^{-1} = T(R)$.

Case 2. $I \neq 0$. By [4, Theorem 25.10], we have $H^{-1} = I^{-1} \times E \neq K \times E = T(R)$ since $I^{-1} \neq K$.

Hence, the ideals $H$ of $R$ such that $H^{-1} = J^{-1}(= T(R))$ have the form $H = 0 \times E''$, where $E''$ is a $D$-submodule of $E$, which is not projective. Therefore, $R$ is not $v$-hereditary, as desired.

References


Received: July, 2010