(2,3,t)-Generations for the Conway group $Co_3$

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Abstract

A group $G$ is said to be $(2,3,t)$-generated if it can be generated by two elements $x$ and $y$ such that $o(x) = 2$, $o(y) = 3$ and $o(xy) = t$. In the present article, we investigate $(2,3,t)$-generations for the Conway's smallest sporadic simple groups $Co_3$, where $t$ is a divisor of the order of $|Co_3|$.

Mathematics Subject Classification: 20D08; 20F05

Keywords: Conway group $Co_3$, sporadic group, simple group, generation

1 Introduction and Preliminaries

A group $G$ is said to be $(2,3)$-generated if it can be generated by an involution $x$ and an element $y$ of order 3. If $o(xy) = t$, we also say that $G$ is $(2,3,t)$-generated. The $(2,3)$-generation problem has attracted a vide attention of group theorists. One reason is that $(2,3)$-generated groups are homomorphic images of the modular group $PSL(2,Z)$, which is the free product of two cyclic groups of order two and three. The connection with Hurwitz groups and Riemann surfaces also play a role. Recall that a $(2,3,7)$-generated group $G$ which gives rise to compact Riemann surface of genus greater than 2 with automorphism group of maximal order, is called Hurwitz group.

Recently, the study of the Conway groups has received considerable amount of attention. Moori in [22] determined the $(2,3,p)$-generations of the smallest Fischer group $F_{22}$. In [16], Ganief and Moori established $(2,3,t)$-generations of the third Janko group $J_3$. More recently, in a series of papers, the authors
established the (2,3,t)-generations for the sporadic groups $H_e$, $HS$, $J_1$ and $J_2$ and the generation by conjugate elements of the groups $HS$, $McL$, $Co_1$, $Co_2$, $Co_3$, $J_1$, $J_2$, $J_3$, $J_4$, $Ly$ and $O'Nan$. See for instance $[1, 2, 3, 4, 5, 6, 8, 9]$.

The present paper is devoted to the study of $(2, 3, t)$-generations of the Conway’s sporadic simple group $Co_3$, where $t$ is a divisor of $|Co_3|$. For more information regarding the study of $(2, 3, t)$-generations as well as the computational techniques, the reader is referred to $[1, 2, 4, 16]$, and $[22]$.

Throughout this paper our notation is standard and taken mainly from $[2, 4, 16]$ and $[22]$. In particular, for a finite group $G$ with $C_1, C_2, ..., C_k$ conjugacy classes of its elements and $g_k$ a fixed representative of $C_k$, we denote $\Delta(G) = \Delta_G(C_1, C_2, ..., C_k)$ the number of distinct tuples $(g_1, g_2, ..., g_{k-1})$ with $g_i \in C_i$ such that $g_1g_2...g_{k-1} = g_k$. It is well known that $\Delta_G(C_1, C_2, ..., C_k)$ is structure constant for the conjugacy classes $C_1, C_2, ..., C_k$ and can easily be computed from the character table of $G$ (see $[18]$, p.45) by the following formula

$$\Delta_G(C_1, C_2, ..., C_k) = \frac{|C_1||C_2|...|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(g_1)\chi_i(g_2)...\chi_i(g_{k-1})\chi_i(g_k)}{|\chi_i(1G)|^{k-2}}$$

where $\chi_1, \chi_2, ..., \chi_m$ are the irreducible complex characters of $G$. Further let $\Delta^*(G) = \Delta^*_G(C_1, C_2, ..., C_k)$ denote the number of distinct tuples $(g_1, g_2, ..., g_{k-1})$ with $g_i \in C_i$ and $g_1g_2...g_{k-1} = g_k$ such that $G = \langle g_1, g_2, ..., g_{k-1} \rangle$. If $\Delta^*_G(C_1, C_2, ..., C_k) > 0$, then we say that $G$ is $(C_1, C_2, ..., C_k)$-generated. If $H$ any subgroup of $G$ containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1, C_2, ..., C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2, ..., g_{k-1}) \in (C_1 \times C_2 \times ... \times C_{k-1})$ such that $g_1g_2...g_{k-1} = g_k$ and $\langle g_1, g_2, ..., g_{k-1} \rangle \leq H$ where $\Sigma_H(C_1, C_2, ..., C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2, ..., c_k)$ of $H$ over all $H$-conjugacy classes $c_1, c_2, ..., c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1 \leq i \leq k - 1$.

For the description of the conjugacy classes, the character tables, permutation characters and information on the maximal subgroups readers are referred to ATLAS $[12]$. A general conjugacy class of elements of order $n$ in $G$ is denoted by $nX$. For example $2A$ represents the first conjugacy class of involutions in a group $G$. In most instances it will be clear from the context to which conjugacy classes we are referring. Thus we shall often suppress the conjugacy classes, using $\Delta(G)$ and $\Delta^*(G)$ as abbreviated notation for $\Delta_G(lX, mY, nZ)$ and $\Delta^*_G(lX, mY, nZ)$, respectively.

The following results in certain situations are very effective at establishing non-generations.

**Theorem 1.1** (Scott’s Theorem $[25]$) Let $x_1, x_2, ..., x_m$ be elements generating a group $G$ with $x_1x_2...x_n = 1_G$, and $V$ be an irreducible module for $G$ of dimension $n \geq 2$. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i \rangle$ on $V$, and let $d_i$ is the codimension of $V/C_V(x_i)$. Then $d_1 + d_2 + ... + d_m \geq 2n$. 
Lemma 1.2 ([11]) Let $G$ be a finite centerless group and suppose $lX$, $mY$, $nZ$ are $G$-conjugacy classes for which $\Delta^*(G) = \Delta_G(lX, mY, nZ) < |C_G(z)|$, $z \in nZ$. Then $\Delta^*(G) = 0$ and therefore $G$ is not $(lX, mY, nZ)$-generated.

The following result will be crucial in determining generating triples.

Theorem 1.3 (Moori [22]) Let $G$ be a finite group and $H$ a subgroup of $G$ containing a fixed element $x$ such that $\gcd(o(x), [N_G(H):H]) = 1$. Then the number $h$ of conjugates of $H$ containing $x$ is $\chi_H(x)$, where $\chi_H$ is the permutation character of $G$ with action on the conjugates of $H$. In particular,

$$h = \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes that fuse to the $G$-class $[x]_G$.

2 Main Results

The Conway group $Co_3$ is a sporadic simple group of order $2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$ with 14 conjugacy classes of its maximal subgroups. We list the maximal subgroups of $Co_3$ in Table I as listed in the GAP. The group $Co_3$ has 42 conjugacy classes of its elements. It has two conjugacy classes of involutions, namely $2A$ and $2B$ and three class of elements of order three as represented in the ATLAS by $3A, 3B$ and $3C$.

Next, we investigate the $(2, 3, t)$-generations of the Conway’s smallest sporadic simple group $Co_3$ where $t$ is a divisor of $|Co_3|$. It is a well known fact that if $G$ is $(2, 3, t)$-generated simple group, then $1/2 + 1/3 + 1/t < 1$. It follows that in $(2, 3, t)$-generations of the Conway group $Co_3$, we only need to consider $t \in K$ where $K = \{7, 8, 9, 10, 11, 12, 15, 18, 20, 21, 22, 23, 24, 30\}$. However, the case when $t \in K$ is prime has already been investigated by Ganief and Moori in [17], so we only need to consider the cases when $t$ is not prime and $t \in K$.

The group $Co_3$ acts on a 23-dimensional irreducible complex module $V$. Let $d_{nX} = \dim(V/C_V(nX))$, the co-dimension of the fix space (in $V$) of a representative in $nX$. Using the character table of $Co_3$ we list in Table II, the values of $d_{nX}$, for the conjugacy classe $nX$.

<table>
<thead>
<tr>
<th>Group</th>
<th>Order</th>
<th>Group</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1 \cong McL.2$</td>
<td>1796256000</td>
<td>$H_2 \cong HS$</td>
<td>44352000</td>
</tr>
<tr>
<td>$H_3 \cong U_4(3).(2^2)$</td>
<td>13063680</td>
<td>$H_4 \cong M_{23}$</td>
<td>10200960</td>
</tr>
<tr>
<td>$H_5 \cong 3^5:(2 \times M_{11})$</td>
<td>3849120</td>
<td>$H_6 \cong 2.S_6(2)$</td>
<td>2903040</td>
</tr>
<tr>
<td>$H_7 \cong U_3(5).S_3$</td>
<td>756000</td>
<td>$H_8 \cong 3^{1+4}.A_6$</td>
<td>699840</td>
</tr>
<tr>
<td>$H_9 \cong 2^4.A_8$</td>
<td>322560</td>
<td>$H_{10} \cong L_3(4):D_{12}$</td>
<td>241920</td>
</tr>
<tr>
<td>$H_{11} \cong 2 \times M_{12}$</td>
<td>190080</td>
<td>$H_{12} \cong 2^2.(27.3^2).S_3$</td>
<td>27648</td>
</tr>
<tr>
<td>$H_{13} \cong S_3 \times L_2(8):3$</td>
<td>9072</td>
<td>$H_{14} \cong A_4 \times S_5$</td>
<td>1440</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>TABLE II</th>
<th>The co-dimensions $d_{nX} = \dim(V/C_V(nX))$</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>$d_{2A}$</td>
</tr>
<tr>
<td></td>
<td>8</td>
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<tr>
<td></td>
<td>$d_{8C}$</td>
</tr>
<tr>
<td></td>
<td>18</td>
</tr>
</tbody>
</table>

In this section we suppose that $X \in \{A, B\}$ and $Y \in \{A, B, C\}$.

**Lemma 2.1** The Conway’s group $\text{Co}_3$ is $(2, 3, 7)-, (2, 3, 11)-, (2, 3, 23)-$generated.

**Proof.** See Ganief and Moori [17]. □

**Lemma 2.2** The sporadic simple group $\text{Co}_3$ is $(2X, 3Y, nZ)$-generated where $Z \in \{A, B\}$ if and only if $Y = C$ and $n \in \{15, 20, 22\}$

**Proof.** First we consider the case $n = 15$. Using algebraic constants of $\text{Co}_3$ we see that $\Delta_{\text{Co}_3}(2A, 3A, 15A) = \Delta_{\text{Co}_3}(2A, 3B, 15A) = \Delta_{\text{Co}_3}(2A, 3A, 15B) = 0$, and non-generation of $\text{Co}_3$ by these triples follows. Further since $\Delta_{\text{Co}_3}(2A, 3B, 15B) = 5 < 15 = |C_{\text{Co}_3}(15A)|$ by applying Lemma 1.2 we obtain $\Delta^*_{\text{Co}_3}(2A, 3B, 15B) = 0$. Thus, $\text{Co}_3$ is not generated by the triple $(2A, 3B, 15B)$.

For the triples $(2A, 3C, 15B)$ and $(2B, 3B, 15B)$ we compute $d_{2A} + d_{3C} + d_{15B} < 2 \times 22$ and $d_{2B} + d_{3B} + d_{15B} < 2 \times 22$ (see TABLE II). Now an application of Scott’s Theorem (Theorem 1.1) shows that $\text{Co}_3$ is not $(2A, 3C, 15B)$-, and $(2B, 3B, 15B)$-generated.

For the triple $(2B, 3C, 15A)$, we observe that again, up to isomorphism, $H_7$ and $H_8$ are the only maximal subgroups of $\text{Co}_3$ that contain $(2B, 3C, 15A)$-generated subgroups, up to isomorphism, are $H_7$ and $H_8$. However, in each case we compute

$$\Sigma_{H_7}(2B, 3C, 15A) = 0 = \Sigma_{H_8}(2B, 3C, 15A)$$

Hence

$$\Delta^*_{\text{Co}_3}(2B, 3C, 15A) = \Delta_{\text{Co}_3}(2B, 3C, 15A) = 300 > 0,$$

proving that $\text{Co}_3$ is $(2B, 3C, 15A)$-generated.

For the triple $(2A, 3C, 15A)$, we observe that again, up to isomorphism, $H_7$ and $H_8$ are the only maximal subgroups of $\text{Co}_3$ which have non-empty intersection with classes in the triple $(2A, 3C, 15A)$. Further, an element of
order 15 is contained in a unique conjugate class of $H_7$. We calculate that $\Sigma_{H_7}(2A, 3C, 15A) = 0$ and we get

$$ \Delta^*_{Co_3}(2A, 3C, 15A) \leq \Delta_{Co_3}(2A, 3C, 15A) - \Sigma_{H_7}(2A, 3C, 15A) = 30 - 30 = 0 < |C_{Co_3}(15A)|. $$

Again by applying Lemma 2.2, we obtain $\Delta^*_{Co_3}(2A, 3C, 15A) = 0$, which shows non-generation of $Co_3$ by this triple.

The only maximal subgroups of $Co_3$ having non-empty intersection with the classes $2B$, $3A$ and $15B$, up to isomorphisms, is $H_5 \cong 3^5:(2 \times M_{11})$. However the structure constant $\Sigma_{H_5}(2B, 3A, 15B) = 0$. Hence

$$ \Delta^*_{Co_3}(2B, 3A, 15B) = \Delta_{Co_3}(2B, 3A, 15B) = 15. $$

This shows that $Co_3$ is $(2B, 3A, 15B)$-generated.

Next we consider the triples $(2B, 3A, 15A)$. Up to isomorphism, $H_1 \cong McL:2$, $H_6 \cong 2.S_6(2)$, $H_7 \cong U_3(5):S_3$ and $H_8 \cong 3^{1+4}:4S_6$ are the maximal subgroups that may contain $(2B, 3A, 15A)$-generated subgroups. But, in each case, we compute that for $i \in \{1, 6, 7, 8\}$, $\Sigma_{H_i}(2B, 3A, 15A) = 0$ and so

$$ \Delta^*_{Co_3}(2B, 3A, 15A) = \Delta_{Co_3}(2B, 3A, 15A) = 30, $$

demonstrating generation of $Co_3$ by the triples $(2B, 3A, 15A)$.

Now consider the triple $(2B, 3B, 15A)$. Up to isomorphism, $H_1 \cong McL:2$, $H_6 \cong 2.S_6(2)$, $H_7 \cong U_3(5):S_3$ and $H_8 \cong 3^{1+4}:4S_6$ are the maximal subgroups of $Co_3$ which have non-empty intersection with classes in the triple $(2B, 3B, 15A)$. Further, an element of order 15 is contained in a unique conjugate class of $H_6$. We calculate that $\Sigma_{H_6}(2B, 3B, 15A) = 30$, and

$$ \Sigma_{H_1}(2B, 3B, 15A) = \Sigma_{H_7}(2B, 3B, 15A) = \Sigma_{H_8}(2B, 3B, 15A) = 0. $$

Hence

$$ \Delta^*_{Co_3}(2B, 3B, 15A) = \Delta_{Co_3}(2B, 3B, 15A) - \Sigma_{H_6}(2B, 3B, 15A) = 30 - 30 = 0 < |C_{Co_3}(15A)|. $$

Again by applying Lemma 2.2, we obtain $\Delta^*_{Co_3}(2B, 3B, 15A) = 0$, which shows non-generation of $Co_3$ by this triple.

Finally, we consider the case $(2B, 3C, 15B)$. Up to isomorphism, the maximal subgroups of $Co_3$ having non-empty intersection with classes in the triple $(2B, 3C, 15B)$ are $H_5 \cong 3^5:(2 \times M_{11})$, $H_{10} \cong L_3(4):D_{12}$ and $H_{14} \cong (A_4 \times S_5)$. 

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An element of order 15 in $\text{Co}_3$ is contained in two conjugate copies of $H_{14}$. We calculate that $\Sigma_{H_5}(2B, 3C, 15B) = 0 = \Sigma_{H_{11}}(2B, 3C, 15B) = 0$, and $\Sigma_{H_{14}}(2B, 3C, 15B) = 15$. We have

$$\Delta^*_{\text{Co}_3}(2B, 3C, 15B) = \Delta_{\text{Co}_3}(2B, 3C, 15B) - 2\Sigma_{H_{14}}(2B, 3C, 15B) = 540 - 2(15) > 0.$$ 

Thus $(2B, 3C, 15B)$ is a generating triple for $\text{Co}_3$.

Next, we consider the cases when $n = 20, 22$. Let $20AB$ represents the class $20A$ or $20B$ and $22AB$ represents the class $22A$ or $22B$. We compute the structure constants

$$\Delta_{\text{Co}_3}(2X, 3Y, 20AB) = 0 = \Delta_{\text{Co}_3}(2X, 3Y, 22AB)$$

where $(X, Y) \in \{(A, A), (A, B), (B, A)\}$ and non-generation of $\text{Co}_3$ by these triples follows immediately.

Now we consider the triple $(2A, 3C, 20AB)$. From the list of maximal subgroups of $\text{Co}_3$ (Table I), we observe that, up to isomorphisms, $H_7$ and $H_8$ are the only maximal subgroups of $\text{Co}_3$ with non-empty intersection with each of the classes in this triple. However, $\Sigma(H_7) = 0 = \Sigma(H_8)$. Hence $\text{Co}_3$ is $(2A, 3C, 20AB)$-generated since $\Delta^*(\text{Co}_3) = \Delta(\text{Co}_3) = 60$.

The only maximal subgroups of $\text{Co}_3$ that may contain $(2B, 3C, 20AB)$-generated proper subgroups, up to isomorphisms, are isomorphic to $H_7$ and $H_8$. We calculate $\Delta_{\text{Co}_3}(2B, 3C, 20AB) = 600$, $\Sigma_{H_7}(2B, 3C, 20AB) = 100$ and $\Sigma_{H_8}(2B, 3C, 20AB) = 60$. Thus,

$$\Delta^*(\text{Co}_3) = \Delta(\text{Co}_3) - \Sigma(H_7) - \Sigma(H_8) = 600 - 100 - 60 > 0,$$

proving generation of $\text{Co}_3$ by this triple.

For the triple $(2A, 3C, 22AB)$, we compute structure constant $\Delta(\text{Co}_3) = 44$ and amongst the maximal subgroup of $\text{Co}_3$ having non-empty intersection with all the classes in this triple, up to isomorphism, are $H_5$ and $H_{11}$. However, $\Sigma(H_5) = 0 = \Sigma(H_{11})$ and we obtain $\Delta^*(\text{Co}_3) = \Delta(\text{Co}_3) = 44$, proving that $\text{Co}_3$ is $(2A, 3C, 22B)$-generated.

Finally consider the triple $(2B, 3C, 22AB)$. $H_5$ and $H_{11}$ are the only maximal subgroups of $\text{Co}_3$ that may contain $(2B, 3C, 22AB)$-generated proper
subgroups. For this triple we calculate \( \Delta(Co_3) = 638, \Sigma(H_5) = 66 \) and \( \Sigma(H_{11}) = 22 \). Hence

\[
\Delta^*(Co_3) = \Delta(Co_3) - \Sigma(H_5) - \Sigma(H_{11}) = 638 - 66 - 22 > 0.
\]

Therefore, \((2B, 3C, 22AB)\) is a generating triple for \(Co_3\) and this completes the proof.

\[\square\]

**Lemma 2.3** The group \(Co_3\) is \((2X, 3Y, 8Z)\)-generated for \(Z \in \{A, B, C\}\), if and only if \((X, Y, Z) = (B, C, C)\).

**Proof.** The non-generation of \(Co_3\) by the triples \((2A, 3A, 8A), (2A, 3A, 8B), (2A, 3B, 8B), (2A, 3C, 8B)\) and \((2A, 3A, 8C)\) follows trivially since the structure constant for each triple is zero. Further we have

\[
\Delta_{Co_3}(2A, 3B, 8A) < |C_{Co_3}(8A)|,
\Delta_{Co_3}(2A, 3C, 8A) < |C_{Co_3}(8A)|,
\Delta_{Co_3}(2B, 3A, 8A) < |C_{Co_3}(8A)|,
\Delta_{Co_3}(2B, 3A, 8B) < |C_{Co_3}(8B)|,
\Delta_{Co_3}(2B, 3A, 8C) < |C_{Co_3}(8C)|.
\]

Thus, an application of Lemma 1.2 shows that the group \(Co_3\) is not \((2A, 3B, 8A)-, (2A, 3C, 8A)-, (2B, 3A, 8A)-, (2B, 3A, 8B)- or (2B, 3A, 8C)-generated. The triples \((2B, 3C, 8A)\) and \((2B, 3C, 8B)\) violate the Ree’s theorem (see [23]), and hence are not generating triples for \(Co_3\).

Let \(Q = \{(2B, 3B, 8A), (2B, 3B, 8B), (2A, 3C, 8C), (2B, 3B, 8C)\}\). It is clear from Table II, that all the triples in the set \(Q\) violate Scott’s theorem and hence do not generate the group \(Co_3\).

Finally we consider the triple \((2B, 3C, 8C)\). In order to show that \(Co_3\) is not \((2B, 3C, 8C)\)-generated, we construct the group \(Co_3\) using its standard generators given in [26, 27]. The group \(Co_3\) has a 22-dimensional irreducible representation over \(GF(2)\). Using this representation we generate \(Co_3 = \langle a, b \rangle\) where \(a\) and \(b\) are \(22 \times 22\) matrices over \(GF(2)\) with \(a \in 3A, b \in 4A\). Using this representation we produce \(x, y\) and \(z\) such that \(x \in 2B, y \in 3C\) and \(z = xy \in 8C\). Let \(H = \langle x, y \rangle\) then \(H \cong U_3(5):S_3\). Now using \(GAP\) programme given in [7] we showed that \(Co_3\) is not \((2B, 3C, 8C)\)-generated.

**Lemma 2.4** The Conway smallest group \(Co_3\) is \((2X, 3Y, nZ)\)-generated, for \(nZ \in \{9A, 9B, 18A, 24A, 24B, 30A\}\), if and only if the ordered pair \((X, Y) = (B, C)\).

**Proof.** Using the algebra constants, we see that for each \(nZ \in T\), we have \(\Delta_{Co_3}(2X, 3Y, nZ) < |C_{Co_3}(nZ)|\) except \((X, Y, Z) \notin T = \{(2B, 3C, 9B), (2B, 3C, 18A), (2B, 3C, 24A), (2B, 3C, 24B), (2A, 3C, 30A), (2B, 3B, 30A), (2B, 3C, 30A)\}\).
Applying Lemma 1.2 we obtain $\Delta^*_{Co_3}(2X, 3Y, nZ) = 0$ for each triple except $(X, Y, Z) \in T$, proving that the group $Co_3$ is not $(2X, 3Y, nZ)$-generated except for $(X, Y, Z) \notin T$.

Next consider the triples $(2B, 3C, 9B)$ and $(2B, 3C, 18A)$. The maximal subgroups of $Co_3$ which have non-empty intersection with the classes in these triples, up to isomorphism, are $H_5$, $H_8$ and $H_{13}$. Further we compute that $\Sigma_{H_5}(2B, 3C, 9B) = 0 = \Sigma_{H_8}(2B, 3C, 9B)$ and $\Sigma_{H_{13}}(2B, 3C, 9B) = 9$. Also, $\Sigma_{H_5}(2B, 3C, 18A) = 0 = \Sigma_{H_8}(2B, 3C, 18A)$ and $\Sigma_{H_{13}}(2B, 3C, 18A) = 18$. Our calculations give

$$\Delta^*_{Co_3}(2B, 3C, 9B) = \Delta_{Co_3}(2B, 3C, 9B) - 9\Sigma_{H_{13}}(2B, 3C, 9B)$$

$$= 810 - 9(9) > 0.$$  

$$\Delta^*_{Co_3}(2B, 3C, 18A) = \Delta_{Co_3}(2B, 3C, 18A) - 3\Sigma_{H_{13}}(2B, 3C, 18A)$$

$$= 648 - 3(18) > 0.$$  

proving that $Co_3$ is $(2B, 3C, 9B)$-$(2B, 3C, 18A)$-generated.

Now we consider the case when $nZ = 24AB$ where $24AB$ represents the class $24A$ or $24B$. We calculate $\Delta_{Co_3}(2B, 3C, 24AB) = 480$. The $(2B, 3C, 24AB)$-generated proper subgroups of $Co_3$ are contained in the maximal subgroups $H_7$, $H_8$ or $H_{12}$. However, our calculations show that $\Sigma_i(2B, 3C, 24AB) = 0$ for $i = \{7, 8, 12\}$ and generation of the triple $(2B, 3C, 24AB)$ follows since $\Delta^*(Co_3) = \Delta(Co_3) = 480$.

For the triples $(2A, 3C, 30A)$ and $(2B, 3B, 30A)$, we observe that the maximal subgroups of $Co_3$ with non-empty intersection with the classes in these triples are, up to isomorphism, $H_1$, $H_6$, $H_7$ and $H_8$. We calculate that $\Sigma_{H_1}(2B, 3B, 30A) = \Sigma_{H_6}(2B, 3B, 30A) = \Sigma_{H_8}(2A, 3C, 30A) = 0$ and $\Sigma_{H_6}(2B, 3B, 30A) = 30 = \Sigma_{H_8}(2A, 3C, 30A)$. Hence

$$\Delta^*_{Co_3}(2A, 3C, 30A) = \Delta_{Co_3}(2A, 3C, 30A) - \Sigma_{H_7}(2A, 3C, 30A)$$

$$= 30 - 30 < |C_{Co_3}(30A)|.$$  

$$\Delta^*_{Co_3}(2B, 3B, 30A) = \Delta_{Co_3}(2B, 3B, 30A) - \Sigma_{H_6}(2B, 3B, 30A)$$

$$= 30 - 30 < |C_{Co_3}(30A)|.$$  

proving non-generation of $Co_3$ by the triples $(2A, 3C, 30A)$ and $(2B, 3B, 30A)$.

Finally, we consider the remaining case $(2B, 3C, 30A)$. The maximal subgroups of $Co_3$ that may contain $(2B, 3C, 30A)$-generated proper subgroups, up to isomorphism, are $H_7$ and $H_8$. Since $\Sigma(H_7) = \Sigma(H_8) = 0$ the generation of $Co_3$ by the triple $(2B, 3C, 30A)$ follows as $\Delta^*(Co_3) = \Delta(Co_3) = 600$. This completes the proof.
Lemma 2.5 The Conway’s sporadic simple group $Co_3$ is $(2X, 3Y, 10Z)$-generated, where $Z \in \{A, B\}$, if and only if the ordered pair $(X, Y) = (B, C)$.

Proof. The non-generation of the triples $(2A, 3A, 10A), (2A, 3B, 10A), (2A, 3C, 10A), (2B, 3A, 10A), (2B, 3A, 10A), (2A, 3A, 10B), (2A, 3B, 10B)$ and $(2B, 3A, 10B)$ follows immediately since the structure constant for each triple in $Co_3$ is zero. Further since
\[
\Delta_{Co_3}(2B, 3B, 10A) = 30 < |Co_3(10A)|
\]
\[
\Delta_{Co_3}(2A, 3C, 10B) = 20 < |Co_3(10B)|,
\]
it follows from Lemma 1.2 that $Co_3$ is not $(2B, 3B, 10A)$-, $(2A, 3C, 10B)$-generated. For the triple $(2B, 3B, 10A)$, we obtained from Table II,
\[
d_{2B} + d_{3B} + d_{10A} = 12 + 12 + 20 = 44 < 46
\]
and hence by Scott’s Theorem (Theorem 1.1), $(2B, 3B, 10A)$ is a non-generating triple for $Co_3$.

Next, we consider the triple $(2B, 3C, 10A)$. From the maximal subgroups of $Co_3$ (Table I), we observe that, up to isomorphism, $H_7$ and $H_8$ are the only maximal subgroups of $Co_3$ that may contain $(2B, 3C, 10A)$-generated proper subgroups. However, we calculate $\Sigma(H_7) = 0 = \Sigma(H_8)$. Therefore, $\Delta^*(Co_3) = \Delta(Co_3) = 600$, proving the generation of $Co_3$ by the triple $(2B, 3C, 10A)$.

Finally consider the triple $(2B, 3C, 10B)$. In this case we have $\Delta(Co_3) = 485$ and $(2B, 3C, 10A)$-generated proper subgroups of $Co_3$ are contained in the only maximal subgroups isomorphic to $H_5, H_7, H_{10}, H_{11}$ and $H_{14}$. Our calculations give,
\[
\Delta^*(Co_3) \geq \Delta(Co_3) - 2\Sigma(H_5) - 2\Sigma(H_7) - 2\Sigma(H_{10}) - 3\Sigma(M_{11}) - \Sigma(M_{14})
\]
\[
= 485 - 2(90) - 2(100) - 2(30) - 3(5) - 5 > 0
\]
proving the generation of $Co_3$ by this triple $(2B, 3C, 10B)$. This completes the proof.

Lemma 2.6 The sporadic group $Co_3$ is $(2X, 3Y, 12Z)$-generated for $Z \in \{A, B, C\}$ if and only if $(X, Y, Z) \in \{(A, C, B), (B, C, C)\}$.

Proof. Set $T = \{(2A, 3A, 12A), (2A, 3B, 12A), (2A, 3A, 12B), (2A, 3B, 12B), (2B, 3A, 12B), (2A, 3C, 12C), (2A, 3B, 12C), (2A, 3A, 12C), (2B, 3A, 12C)\}$. For each triple $(lX, mY, tZ) \in T$ we obtain
\[
\Delta_{Co_3}(lX, mY, tZ) < |Co_3(tZ)|.
\]
Now, an application of Lemma 1.2 shows that $Co_3$ is not $(lX, mY, tZ)$-generated for all $tZ \in T$.

Using character table of $Co_3$ we compute the values of $d_{nX}$ for conjugacy class $nX$ (see Table II)

- $d_{2A} + d_{3C} + d_{12B} = 8 + 16 + 20 = 44 < 46$
- $d_{2B} + d_{3B} + d_{12B} = 12 + 12 + 20 = 44 < 46$
- $d_{2B} + d_{3B} + d_{12C} = 12 + 12 + 20 = 44 < 46$.

Clearly the triples $(2A, 3C, 12B)$, $(2B, 3B, 12B)$ and $(2B, 3B, 12C)$ violate the Scott’s Theorem and therefore are not generating triples for $Co_3$.

For the triple $(2B, 3C, 12A)$, the maximal subgroups of $Co_3$ having non-empty intersection with the classes in this triple, up to isomorphic, are $H_5$, $H_8$ and $H_{12}$. We compute $\Sigma(M_5) = 48$ and $\Sigma(M_8) = 0 = \Sigma(M_{12})$. Thus,

$$\Delta^*(Co_3) \leq \Delta(Co_3) - 4\Sigma(M_5) = 192 - 4 \times 48 < |C_{Co_3}(12A)| = 144.$$ 

Hence, by Lemma 1.2, the group $Co_3$ is not $(2B, 3C, 12A)$-generated.

Next we consider the triple $(2B, 3B, 12B)$. For this triple we have $\Delta(Co_3) = 576$ and maximal subgroups of $Co_3$ containing proper $(2B, 3B, 12B)$-generated proper subgroups are, up to isomorphism, $H_5$, $H_7$, $H_8$ and $H_{12}$. However, we see that

$$\Sigma(H_5) = \Sigma(H_7) = \Sigma(H_8) = \Sigma(H_{12}) = 0,$$

and thus this triple is a generating triple for $Co_3$ since $\Delta^*(Co_3) = \Delta(Co_3)$.

Finally, we consider the triple $(2B, 3C, 12C)$. The maximal subgroups of $Co_3$ with non-empty intersection with all the classes in this triple, up to isomorphism, are $H_5$, $H_7$, $H_8$, $H_{10}$, $H_{12}$ and $H_{14}$. We calculate $\Delta(Co_3) = 696$, $\Sigma(H_5) = 48$, $\Sigma(H_7) = 104$, $\Sigma(H_8) = 108$, $\Sigma(H_{10}) = 12$, $\Sigma(H_{12}) = 40$ and $\Sigma(H_{14}) = 16$. Thus, we obtain

$$\Delta^*(Co_3) \geq 696 - 48 - 104 - 108 - 3(12) - 6(40) - 6(16) > 0.$$ 

Therefore, $Co_3$ is $(2B, 3C, 12C)$-generated and this completes the Lemma.

**Lemma 2.7** The Conway group $Co_3$ is $(2X, 3Y, 21A)$-generated if and only if $Y = C$. 
(2,3,t)-generations for $\text{Co}_3$

**Proof.** Since $\Delta_{\text{Co}_3}(2X, 3Y, 21A) = 0$ for $Y \in \{A, B\}$. Therefore, $\Delta^*_{\text{Co}_3}(2X, 3Y, 21A) = 0$ and $\text{Co}_3$ is not $(2X, 3A, 21A)$-, $(2X, 3B, 21A)$-generated.

Next consider the case $(2B, 3C, 21A)$. The structure constants $\Delta_{\text{Co}_3}(2B, 3C, 21A) = 63$. The only maximal subgroups of $\text{Co}_3$ which have order divisible by $2 \times 3 \times 21$ and have non-empty intersection with the classes $2B$, $3C$ and $21A$ are, up to isomorphism, $H_7 \cong U_3(5)$, $H_{10} \cong L_3(4):D_{12}$ and $H_{13} \cong S_3 \times L_2(8):3$. Direct computation on GAP shows that $\Sigma_{H_7}(2B, 3C, 21A) = 0 = \Sigma_{H_{10}}(2B, 3C, 21A)$ and $\Sigma_{H_{13}}(2B, 3C, 21A) = 7$. Since an element of order $21$ in $\text{Co}_3$ is contained in precisely unique conjugate class of $P$, it then follows that $\Delta^*_{\text{Co}_3}(2B, 3C, 21A) \geq 532 - 1(7) > 0$. Hence $\text{Co}_3$ is $(2B, 3C, 21A)$-generated. This completes the proof.

**Theorem 2.8** The Conway’s sporadic simple group $\text{Co}_3$ is $(2,3,t)$-generated except for $t = 8$.

**Proof.** The proof follows from the Lemma 2.1 - 2.7. \hfill $\square$

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**References**


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