Further Results on Graded Prime Submodules

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Abstract

Let $G$ be a group, $R$ be a $G$ - graded ring and $M$ be a $G$ - graded $R$ - module, i.e., $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. In this paper, we introduce several results concerning graded prime submodules.

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Introduction

A commutative ring $R$ with unity is a $G$ - graded ring if there exist additive subgroups $R_g$ of $R$ indexed by the elements $g \in G$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by $(R, G)$ and we consider $\text{supp}(R, G) = \{g \in G : R_g \neq 0\}$. The elements of $R_g$ are called homogeneous of degree $g$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Moreover, $R_e$ is a subring of $R$ and $1 \in R_e$. Also, if $r \in R_g$ and $r$ is a unit, then $r^{-1} \in R_{g^{-1}}$. Finally, $h(R) = \bigcup_{g \in G} R_g$. Let $M$ be a left $R$ - module. Then $M$ is a $G$ - graded $R$ - module (in short, $M$ is gr - $R$...
- module) if there exist additive subgroups $M_g$ of $M$ indexed by the elements $g \in G$ such that $M = \bigoplus_{g \in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. We denote this by $(M, G)$ and we consider $\text{supp}(M, G) = \{g \in G : M_g \neq 0\}$. The elements of $M_g$ are called homogeneous of degree $g$. If $x \in M$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $M_g$. Clearly, $M_g$ is an $R_g$-submodule of $M$ for all $g \in G$. Finally, $h(M) = \bigcup_{g \in G} M_g$. For more details, one can look in [1, 2, 6].

In Section 1, we give some basic concepts concerning graded rings and graded modules.

In Section 2, we introduce several results concerning graded prime submodules.

1. Preliminaries

In this section, we give some basic concepts concerning graded rings and graded modules.

Definition 1.1. [7] $(R, G)$ is called first strong if $1 \in R_g R_g^{-1}$ for all $g \in \text{supp}(R, G)$.

Proposition 1.2. [7] Let $R$ be a $G$-graded ring. Then $(R, G)$ is first strong if and only if $\text{supp}(R, G)$ is a subgroup of $G$ and $R_g R_h = R_{gh}$ for all $g, h \in \text{supp}(R, G)$.

Definition 1.3. [10] $(M, G)$ is called first strong if $\text{supp}(R, G)$ is a subgroup of $G$ and $R_g M_h = M_{gh}$ for all $g \in \text{supp}(R, G), h \in G$.

Proposition 1.4. [10] Let $R$ be a $G$-graded ring. Then $(R, G)$ is first strong if and only if every $gr$-$R$-module is first strongly graded.

Definition 1.5. [6] Let $R$ be a $G$-graded ring and $I$ be an ideal of $R$. Then $I$ is called $G$-graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g)$.

Definition 1.6. [6] Let $M$ be a $G$-module and $N$ be an $R$-submodule of $M$. Then $N$ is called $G$-gr-$R$-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$.

Remark 1.7. An ideal (submodule) of a graded ring (graded module) need not be graded. For more details, see [9].

Definition 1.8. [5] Let $M$ be a $gr$-$R$-module. Then $M$ is called graded multiplication module (in short, $gr$-$R$-multiplication module) if for every $gr$-$R$-submodule $N$ of $M$, $N = IM$ for some $gr$-ideal $I$ of $R$.

Definition 1.9. [8] $(R, G)$ is called crossed product over the support if $R_g$ contains a unit for all $g \in \text{supp}(R, G)$. 
Proposition 1.10. [8] If \((R,G)\) is crossed product over the support then \((R,G)\) is first strong.

Proposition 1.11. [8] Let \(R\) be a \(G\) - graded ring. If \(R_e\) is a division ring, then \((R,G)\) is crossed product over the support.

Proposition 1.12. [10] Let \(R\) be a first strongly \(G\) - graded ring and \(M\) be an \(R\) - module. If \(N\) and \(L\) are \(R_e\) - submodules of \(M\), then \(R_g(N \cap L) = R_gN \cap R_gL\) for all \(g \in G\).

2. Graded Prime Submodules

In this section, we introduce several results concerning graded prime submodules. Firstly, we begin with the following result that can be found in [4].

Proposition 2.1. Let \(M\) be a gr - \(R\) - module. If \(N\) is a gr - \(R\) - submodule of \(M\), then \((N : R M) = \{r \in R : rM \subseteq N\}\) is gr - ideal of \(R\).

Definition 2.2. [4] Let \(M\) be a gr - \(R\) - module and \(N \neq M\) be a gr - \(R\) - submodule of \(M\). Then \(N\) is called graded prime \(R\) - submodule of \(M\) if whenever \(r \in h(R)\) and \(m \in h(M)\) with \(rm \in N\), then either \(m \in N\) or \(r \in (N :_R M)\).

Definition 2.3. [9] Let \(I\) be a proper gr - ideal of a graded ring \(R\). Then \(I\) is called graded prime ideal of \(R\) if whenever \(a,b \in h(R)\) with \(ab \in I\), then either \(a \in I\) or \(b \in I\).

Proposition 2.4. Let \(M\) be a gr - \(R\) - module. If \(N\) is a graded prime - \(R\) - submodule of \(M\), then \((N :_R M)\) is graded prime ideal of \(R\).

Proof. By Proposition 2.1, \((N :_R M)\) is graded ideal of \(R\). Let \(a,b \in h(R)\) with \(ab \in (N :_R M)\). Then \(abM \subseteq N\). Since \(N\) is graded prime, \(N \neq M\) and then there exists \(m \in M - N\). Now, \(abm \in abM \subseteq N\) and since \(N\) is graded prime, either \(bm \in N\) or \(a \in (N :_R M)\). If \(bm \in N\), then since \(N\) is graded prime and \(m \notin N\), \(b \in (N :_R M)\). Hence \((N :_R M)\) is graded prime.

Proposition 2.5. Let \(R\) be a first strongly \(G\) - graded ring and \(M\) be a gr - \(R\) - module. If \(N\) is a graded prime - \(R\) - submodule of \(M\), then \(N_g\) is a prime \(R_e\) - submodule of \(M_g\) for all \(g \in \text{supp}(R,G)\).

Proof. Let \(g \in \text{supp}(R,G)\), \(r_e \in R_e\) and \(m_g \in M_g\) such that \(r_em_g \in N_g\). Since \((rm)_g = r_em_g\), \(r \in h(R)\) and \(m \in h(M)\) with \(rm \in N\). Since \(N\) is
graded prime, either \( m \in N \) or \( r \in (N :_R M) \). If \( m \in N \), then \( m_N \in N_g \). If \( r \in (N :_R M) \), then \( r_N \in (N :_R M) = (N :_R M) \cap R_e = (N_e :_R M_e) \) and then \( r_N M_g = r_N g_M e = g_N e_M e \subseteq N_g N_e \subseteq N_g \), i.e., \( r_N \in (N_g :_R M_g) \). Hence \( N_g \) is prime.

**Proposition 2.6.** Suppose \((R, G)\) is crossed product over the support and \( M \) is a gr - \( R \) - module. If \( N \) is a gr - \( R \) - submodule of \( M \) such that \( N_e \) is prime \( R_e \) - submodule of \( M_e \), then \( N_g \) is a prime \( R_e \) - submodule of \( M_g \) for all \( g \in \text{supp}(R, G) \).

**Proof.** Let \( g \in \text{supp}(R, G) \), \( r \in R_e \) and \( m \in M_g \) such that \( rm \in N_g \). Since \((R, G)\) is crossed product over the support, \( R_g^{-1} \) contains a unit, say \( x \) and by Proposition 1.10, \((R, G)\) is first strong. Now, \( r x m = x r m \in R_g^{-1} N_g \subseteq N_e \). Since \( N_e \) is prime, either \( xm \in N_e \) or \( r \in (N_e :_R M_e) \). If \( xm \in N_e \), then \( m = 1 m = x^{-1} x m \in R_g N_e \subseteq N_g \). If \( r \in (N_e :_R M_e) \), then \( r M_e \subseteq N_e \) and then \( r M_g = r R_g M_e = R_g r M_e \subseteq R_g N_e \subseteq N_g \), i.e., \( r \in (N_g :_R M_g) \). Hence \( N_g \) is prime \( R_e \) - submodule.

**Definition 2.7.** [4] Let \( M \) be a gr - \( R \) - module and \( N \) be a gr - \( R \) - submodule of \( M \). Then \( N \) is called graded pure \( R \) - submodule of \( M \) if \( rN = N \cap rM \) for all \( r \in h(R) \).

Atani has proved in [4] that if \( N \) is a graded pure \( R \) - submodule of \( M \) and \( M \) is \( R \) - torsion free, then \( N_g \) is a pure \( R_e \) - submodule of \( M_g \) for all \( g \in G \).

On the other hand, we introduce the following result.

**Proposition 2.8.** Suppose \((R, G)\) is crossed product over the support and \( M \) is a gr - \( R \) - module such that \( N \) is \( R \) - torsion free. Let \( N \) be a gr - \( R \) - submodule of \( M \). If \( N_e \) is pure \( R_e \) - submodule of \( M_e \), then \( N_g \) is a pure \( R_e \) - submodule of \( M_g \) for all \( g \in \text{supp}(R, G) \).

**Proof.** Let \( g \in \text{supp}(R, G) \) and \( r \in R_e \). If \( r = 0 \), then \( r N_g = 0 = N \cap r M_g \).

Suppose \( r \neq 0 \). Clearly, \( r N_g \subseteq N_g \cap r M_g \). Let \( x \in N_g \cap r M_g \). Then \( x = r m \in N_g \) for some \( m \in M_g \). Since \((R, G)\) is crossed product over the support, \( R_g^{-1} \) contains a unit, say \( y \) and by Proposition 1.10, \((R, G)\) is first strong. Now, \( r y m = y r m \in R_g^{-1} (N_g \cap r M_g) \subseteq R_g^{-1} N_g \cap R_g^{-1} r M_g \subseteq N_e \cap r M_e = r N_e \) since \( N_e \) is pure and then \( r y m = r n \) for some \( n \in N_e \), i.e., \( r (y m - n) = 0 \).

Since \( M \) is \( R \) - torsion free and \( r \neq 0 \), \( y m = n \in N_e \) and then \( m = 1 m = y^{-1} y m \in R_g N_g \subseteq N_g \). So, \( x = r m \in r N_g \), i.e., \( N_\cap r M_g \subseteq r N_g \). Hence \( N_g \) is pure \( R_e \) - submodule of \( M_g \).

For \( g \in G \), define \( T(M_g) = \{ m \in M_g : r m = 0 \) for some nonzero \( r \in R_e \} \).

One can prove easily that if \( R_e \) has no zero divisors, then \( T(M_g) \) is an \( R_e \) - submodule of \( M_g \) for all \( g \in G \).
Proposition 2.9. Let $R$ be a $G$-graded ring such that $R_e$ is a division ring. Suppose $M$ is a gr-$R$-module. If $T(M_e)$ is prime $R_e$-submodule of $M_e$, then $T(M_g)$ is a prime $R_e$-submodule of $M_g$ for all $g \in \text{supp}(R,G)$.

Proof. Let $g \in \text{supp}(R,G)$, $r \in R_e$ and $m \in M_g$ such that $rm \in T(M_g)$. Then there exists a nonzero $\alpha \in R_e$ such that $\alpha rm = 0$. Since $R_e$ is division ring, $(R,G)$ is crossed product over the support, and then $R_{g^{-1}}$ contains a unit, say $x$ and by Proposition 1.10, $(R,G)$ is first strong. Now, $rxm \in R_eR_{g^{-1}}M_g = M_e$ and $\alpha rxm = x\alpha rm = 0$ and then $rxm \in T(M_e)$. Since $T(M_e)$ is prime, either $xm \in T(M_e)$ or $rM_e \subseteq T(M_e)$. If $xm \in T(M_e)$, then there exists a nonzero $y \in R_e$ such that $yxm = 0$ and then $ym = yx^{-1}xm = x^{-1}yxm = 0$, i.e., $m \in T(M_g)$. Suppose $rM_e \subseteq T(M_e)$. Let $s \in M_g$. Then $rs \in rR_{g^{-1}}M_g = rM_e \subseteq T(M_e)$ and then there exists a nonzero $t \in R_e$ such that $trs = 0$. So, $trs = trx^{-1}xs = x^{-1}trxs = 0$ and hence $rs \in T(M_g)$, i.e., $rM_g \subseteq T(M_g)$. Hence $T(M_g)$ is prime $R_e$-submodule.

Now, we introduce the concept of graded strongly irreducible submodules.

Definition 2.10. Let $M$ be a gr-$R$-module and $N$ be a gr-$R$-submodule of $M$. Then $N$ is called graded strongly irreducible $R$-submodule of $M$ if whenever $N_1$ and $N_2$ are gr-$R$-submodules of $M$ with $N_1 \cap N_2 \subseteq N$, then either $N_1 \subseteq N$ or $N_2 \subseteq N$.

Proposition 2.11. Let $R$ be a first strongly $G$-graded ring. Suppose $M$ is a gr-$R$-module and $N$ is a gr-$R$-submodule of $M$. If $N$ is graded strongly irreducible, then $N_g$ is a strongly irreducible $R_e$-submodule of $M_g$ for all $g \in G$.

Proof. Let $g \in G$, $N_1$ and $N_2$ be a gr-$R_e$-submodules of $M_g$ with $N_1 \cap N_2 \subseteq N_g$. Then $RN_1$ and $RN_2$ are gr-$R$-submodules of $M$ with $RN_1 \cap RN_2 = R(N_1 \cap N_2) \subseteq RN_g \subseteq RN = N$ (see Proposition 1.12). Since $N$ is strongly irreducible, either $RN_1 \subseteq N$ or $RN_2 \subseteq N$ and then either $(RN_1)_g \subseteq N_g$ or $(RN_2)_g \subseteq N_g$. But $(RN_1)_g = RN_1 \cap M_g = R_eN_1 = N_1$ and similarly, $(RN_2)_g = N_2$. So, either $N_1 \subseteq N_g$ or $N_2 \subseteq N_g$. Hence $N_g$ is strongly irreducible $R_e$-submodule of $M_g$.

Proposition 2.12. Let $R$ be a first strongly $G$-graded ring. Suppose $M$ is a gr-$R$-module and $N$ is a gr-$R$-submodule of $M$. If $N_e$ is strongly irreducible $R_e$-submodule of $M_e$, then $N_g$ is a strongly irreducible $R_e$-submodule of $M_g$ for all $g \in \text{supp}(R,G)$.

Proof. Let $g \in \text{supp}(R,G)$, $N_1$ and $N_2$ be a gr-$R_e$-submodules of $M_g$ with $N_1 \cap N_2 \subseteq N_g$. Then $R_{g^{-1}}N_1$ and $R_{g^{-1}}N_2$ are $R_e$-submodules of $M_e$ with
$R_g^{-1}N_1 \cap R_g^{-1}N_2 = R_g^{-1}(N_1 \cap N_2) \subseteq R_g^{-1}N_g \subseteq N_e$ (see Proposition 1.12). Since $N_e$ is strongly irreducible, either $R_g^{-1}N_1 \subseteq N_e$ or $R_g^{-1}N_2 \subseteq N_e$ and then either $N_1 = R_eN_1 = R_gR_g^{-1}N_1 \subseteq R_gN_g \subseteq N_g$ or similarly, $N_2 \subseteq N_g$. Hence $N_g$ is strongly irreducible $R_e$-submodule of $M_g$. \hfill \Box

**Proposition 2.13.** Let $M$ be a gr-$R$-multiplication module and $N$ be a gr-$R$-submodule of $M$. If $N$ is graded prime, then $N$ is graded strongly irreducible.

**Proof.** Let $N_1$ and $N_2$ be a gr-$R$-submodules of $M$ with $N_1 \cap N_2 \subseteq N$. Since $M$ is graded multiplication, $N_1 = I_1M$ and $N_2 = I_2M$ for some graded ideals $I_1$ and $I_2$ of $R$. Suppose $N_1 \nsubseteq N$ and $N_2 \nsubseteq N$. Then there exist $r_1 \in I_1$, $r_2 \in I_2$ and $m_1, m_2 \in M$ such that $r_1m_1 \notin N$ and $r_2m_2 \notin N$. Now, $r_1r_2m_2 \in N_1 \cap N_2 \subseteq N$. Since $N$ is graded prime and $r_2m_2 \notin N$, $r_1M \subseteq N$ and then $r_1m_1 \in N$ a contradiction. Hence $N$ is graded strongly irreducible. \hfill \Box

**References**


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