p.q. Baer Ring with Generalized Countable Join

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Abstract

Let $R$ be a ring with unity. Let $\alpha$ be an endomorphism of $R$ and $R_R$ be an $\alpha$-compatible module. Then the formal power series ring $R[[x, \alpha]]$ is right p.q. Baer iff $R$ is right p.q. Baer and every countable subset of right semicentral idempotents has a generalized countable join.

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1 Introduction

Throughout this paper, $R$ denotes an associated ring with unity and $M_R$ will stand for right $R$-module. Recall that $R$ is called (quasi) Baer ring if the right annihilator of every (right ideal) non empty subset of $R$ is generated as right ideal by an idempotent of $R$. In [16] Kaplansky introduced Baer rings to abstracts various properties of $AW^*$-algebra and von Neumman algebras. The class of Baer rings includes the von Neumman algebras. Quasi-Baer rings introduced by Clark [8], are used to characterize when a finite dimensional Algebra over a algebraically closed field is isomorphic to a twisted matrix units semi-group algebra. The definition of quasi Baer ring is left-right i.e. a ring $R$ is left (quasi) Baer ring iff $R$ is right (quasi) Baer ring.

As a generalization of quasi-Baer ring, G.F. Birkenmeier, J.Y. Kim, J.K. Park in [3] introduced the concept of principally quasi-Baer rings. A ring $R$ is called right principally quasi-Baer (or right p.q. Baer) if the right annihilator of principal right ideal of $R$ generated by an idempotent similarly left p.q. Baer rings can be defined. A ring $R$, is p.q. Baer if it is right and left p.q. Baer
ring. The class of p.q. Baer ring includes all q. Baer ring, abelian p.p.-ring and bi-regular ring. For more details and examples of right p.q. Baer ring see [3].

Recently many authors studied the ore extension of quasi-Baer ring and their generalization in [3-6]. It has been proved in [4] that the ring $R$ is quasi-Baer iff $R[x]$ is quasi-Baer if and only if $R[[x]]$ is quasi-Baer where $X$ is an arbitrary non-empty set of not-necessarily commuting indeterminates. If $R$ is reduced (i.e. $R$ has no non-zero nilpotent elements). Then $R$ is Baer iff $R[x]$ is Baer iff $R[[x]]$ is Baer [4]. Recall [4], that an idempotent $e \in R$ is left semicentral in $R$ if $eRe = eR$ for all $r \in R$. Equivalently $e^2 = e \in R$ is left semicentral if $eR$ is an ideal of $R$. Since the right annihilator of right ideal is an ideal. We see that in p.q. Baer ring the right annihilator of principal right ideal is generated by left semicentral idempotent. A ring $R$ is right p.q. Baer iff $R[[x]]$ is right p.q. Baer. But it is not equivalent to that $R[[x]]$ is p.q. Baer. In [17], Z. Liu showed that $R$ is p.q. Baer iff $R[[x]]$ is p.q. Baer and any countable family of idempotent has generalized join when all the left semicentral idempotent are central. In [9] Y. Cheng and F.K. Huang, have shown that the ring $R[[R]]$ is right p.q. Baer iff $R$ is right p.q. Baer and every countable subset of right semicentral idempotents has generalized countable join.

From now onwards we always denote the skew power series ring $T = R[[x, \alpha]]$, where $\alpha : R \rightarrow R$ is an endomorphism. The ring $T$ consisting all the power series of form $\sum_{i=0}^{\infty} a_i x_i^i$ where $a_i \in R$, which are multiplied using the distributive law and the ore commutation rule $xa = \alpha(a)x$ for all $a \in R$. In [1] a module $M_R$, an endomorphism $\alpha : R \rightarrow R$ we say that $M_R$ is $\alpha$-compatible if for each $m \in M$, $r \in R$ we have $mr = 0 \iff ma(r) = 0$. In [13] Ebrahim Hashemi showed that let $R$ be a ring with $S_l(R) \subseteq C(R)$ and $\alpha$ be an endomorphism of $R$. Let $R_R$ be an $\alpha$-compatible module. Then $R[[x, \alpha]]$ is right p.q. Baer iff $R$ is right p.q. Baer and any countable family of idempotents of $R$ has generalized join in $I(R)$. In this note we have removed the condition $S_r(R) \subseteq C(R)$ and shown that let $R$ be a ring and $\alpha$ be an endomorphism of $R$ and let $R_R$ be an $\alpha$-compatible module then $R[[x, \alpha]]$ is right p.q. Baer iff $R$ is right p.q. Baer and every countable subset of semicentral idempotent has a generalized countable join.

2 Main Results

Generalized Join 2.1 In [17], Liu defined that the countable family of idempotents $\{e_0, e_1, \ldots\}$ of $R$ is said to have a generalized join $e$ if there exists $e^2 = e \in R$ such that.

(i) $e_i R(1 - e) = 0$

(ii) If $d$ is an idempotent and $e_i R(1 - d) = 0$ then $e R(1 - d) = 0$
Generalized Countable Join 2.2 Let $R$ be a ring with unity and $E = \{e_0, e_1, \ldots \}$ be a countable subset of $S_r(R)$. We say $E$ has a generalized countable join if, given $a \in R$, there exists $e \in S_r(R)$ such that

(i) $e_i e = e_i$ for all $i \in N$

(ii) if $e_i a = e_i$ for all $i \in N$, then $ea = e$.

Lemma 2.3 ([3, Lemma 1.1]) For an idempotent $e \in R$, the following conditions are equivalent

(1) $e \in S_I(R)$
(2) $(1 - e) \in S_r(R)$
(3) $(1 - e)Re = 0$
(4) $R(1 - e)$ is an ideal of $R$

Lemma 2.4 Let $R$ be a ring and if any countable set of idempotent $\{e_0, e_1, e_2, \ldots \} \subseteq S_r(R)$ has generalized countable join then generalize join.

Proof Suppose $E = \{e_0, e_1 \ldots \} \subseteq S_r(R)$ and $e$ is an idempotent which is generalized countable join pf $E$. By hypothesis $e_i e = e_i$ for all $i \in N \Rightarrow e_i (1 - e) = 0$.

Take

\[
e_i R(1 - e) = e_i Re_i (1 - e) \quad \text{for } e_i \in S_r(R)\]

\[
e_i (1 - e) = 0\]

Now only prove condition (ii) Let $d$ be an idempotent of $R$ such that $e_i R(1 - d) = 0$. Then in particular $e_i (1 - d) = 0 \Rightarrow e_i d = e_i$ then by hypothesis $e.d = e \Rightarrow e(1 - d) = 0 \Rightarrow er (1 - d) = ese (1d) = 0$ for all $r \in R \Rightarrow eR(1 - d) = 0$. Therefore $e$ is a generalized join in $E$.

Remark A right ideal $I$ of a ring of $R$ is said to have the intersection of factors property (Simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$. So a ring $R$ is called IFP if and only if any right annihilator is an ideal if and only if any left annihilator is an ideal iff $ab = 0 \Rightarrow aRb = 0$ for $a, b \in R$. Simple computation gives that the reduced rings are IFP and IFP rings are abelian.

Lemma 2.5 Let $R$ is p.q. Baer ring with IFP condition and if any set of idempotents $\{e_0, e_1, e_2, e_3, \ldots \} \subseteq S_r(R)$ has generalized join $e$. Then it is generalized countable join $e$. 
Proof Let $E = \{e_0, e_1, \ldots \} \subseteq S_r(R)$ and $e$ be a generalized join in $E$ then $e_i R(1 - e) = 0 \Rightarrow e_r(1 - e) = 0$ for all $r \in R$ in particular $e_i(1 - e) = 0 \Rightarrow e_i e = e_i$. Now only to show that if $e_i a = e_i$ for $a \in R \Rightarrow e a = e$. Suppose $d$ be an idempotent on $R$ such that

$$e_i d = e_i$$

$$\Rightarrow e_i (1 - d) = 0$$

But $e_i R(1 - d) = 0 \Rightarrow e R(1 - d) = 0$ (by hypothesis)

In particular

$$e (1 - d) = 0$$

$$\Rightarrow e d = e.$$

Again suppose arbitrary $a \in R$ and $R$ is p.q. Baer ring with IFP property then [3, Prop. 1.14] $R$ is Reduced p.p.-ring. Then $(1 - a) = dt$, for some central idempotent $d \in R$ and for some $t \in R$. $r.ann(dt) = r.ann(t) = 0 = l.ann(dt) = l.ann(t)$, by [10, Proposition 2]. But $e_i a = e_i$

$$\Rightarrow e_i (1 - a) = 0$$

$$\Rightarrow e_i dt = 0$$

$$\Rightarrow e dt = 0 \quad \forall \ i \in \mathbb{N}, \ 0, 1, 2, 3, \ldots$$

$$\Rightarrow ed = 0$$

Take

$$e (1 - a) = edt = 0$$

$$\Rightarrow e (1 - a) = 0$$

$$\Rightarrow e a = e$$

Proposition 2.6 Let a ring $R$ has IFP property. Then the following are equivalent

(i) $R$ is right p.q. Baer, $S_r(R) \subseteq C(R)$ and any countable family of idempotents in $R$ has a generalized join.

(ii) $R$ is right p.q. Baer, $S_r(R) \subseteq C(R)$ and the right annihilator of any countably generalized right ideal of $R$ is generated by an idempotent.

(iii) $R$ is right p.q. Baer and any countable subset $S_r(R)$ has generalized countable join.

Proof (i) $\Leftrightarrow$ (ii) [13, Lemma 2.10]

(ii) $\Rightarrow$ (iii) Suppose that $E = \{e_i | i = 1, 2, \ldots \}$ be countable subset of $S_r(R)$. Suppose that $I$ be right ideal countably generated by idempotents $\{e_i | i =
1, 2, . . . }. Then $\text{ann}_R(I) = eR$ for some $e \in S_r(R)$. Let $h = 1 - e$, then $e_i R(1 - h) = 0$
\[\Rightarrow e_i R(1 - h) = 0 \text{ for all } i \in R\]
\[\Rightarrow e_r R(1 - h) = 0 \text{ for all } r \in R\]
\[\Rightarrow e_i (1 - h) = 0 \Rightarrow e_i h = e_i\]

Now only to show condition (ii) i.e $e_i a = e_i \Rightarrow e a = e$ for $a \in R$ since (2) $\Rightarrow$ (1) i.e. $h$ is generalized join. But $R$ is p.q. Baer and satisfies IFP property then by [Lemma 2] $h$ is generalized countable join.

(iii) $\Rightarrow$ (ii) Let $X = \{a_i | i = 1, 2, \ldots\}$ be countable subset of $R$ and $I$ be a right ideal of $R$ generated by $X$. Then for each $a_i \in X$, $r.\text{ann}(a_i R) = e_i R$ for some idempotent $e_i \in S_r(R)$. Let $h$ be generalized countable join of the set \[\{1 - e_i | i = 1, 2, \ldots\} \subseteq S_i(R)\] then by [Lemma 1] $h$ is generalized join of the set \[\{1 - e_i | i = 1, 2, \ldots\}\]. Therefore by [13. Lemma 2.10] $r.\text{ann}_R(\langle X \rangle)$ is generated by an idempotent.

**Definition 2.7** A ring $R$ is $\alpha$-compatible if $ab = 0 \Leftrightarrow a\alpha(b) = 0$ for $a, b \in R$ where $\alpha$ is an endomorphism of $R$.

**Example 2.8 ([14, p. 225])** For a given field $F$.

Let $R = \left\{ (a_n) \in \prod_{n=1}^{\infty} F_n | a_n \text{ is a eventually constant} \right\}$ which is a subring of $\prod_{n=1}^{\infty} F_n$, where $F_n = F$ for $n = 1, 2, 3, \ldots$. Then $R$ is commutative Von Neumann regular ring and hence it is right p.q. Baer. Let $\alpha$ be an identity map on $R$. Then $R$ is $\alpha$-right ring ($\alpha$-compatible). But $R[[x, \alpha]]$ is not right p.q. Baer.

**Theorem 2.9** Let $M$ be an $\alpha$-compatible module and $T = R[[x, \alpha]]$. Then $M[[x]]_T$ is p.q. Baer iff $M_R$ is p.q. Baer and the right annihilator of any countable generated subset of $M$ is generated by an idempotent.

**Proof** See reference [13, Theorem 2.5].

**Theorem 2.10** Let $R$ be a ring with IFP Property and $\alpha : R \rightarrow R$ be an endomorphism of $R$. Let $R_R$ be an $\alpha$-compatible module. Then $R[[x, \alpha]]$ is right p.q. Baer iff $R$ is right p.q. Baer and any countable subset of $S_r(R)$ has generalized countable join.

**Proof** By Theorem 2.9, Proposition 2.6. The result is true.

**Corollary 2.11 ([9, Theorem 5])** Let $R$ be a ring with unity. Then $R[[x]]$ is right p.q. Baer if and only if $R$ is right p.q. Baer and every countable subset of $S_r(R)$ has a generalized countable join.
Corollary 2.12 ([16, Theorem 3]) Let $R$ be a ring such that $S_l(R) \subseteq C(R)$. Then $R[[x]]$ is right p.q. Baer iff $R$ is right p.q. Baer and countable family of idempotents in $R$ has a join.

Corollary 2.13 ([10, Theorem 3]) If $R$ is a ring then $R[[x]]$ is reduced p.p.-ring if and only if $R$ is a reduced p.p.-ring and any countable family of idempotents in $R$ has a join in $C(R)$.

References


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