Modules whose Closed M-Cyclics are Summand

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Abstract

We introduce the concept of CMS modules. A right $R$-module $M$ is called CMS if, every closed $M$-cyclic submodule of $M$ is a direct summand. An example of CMS module which is not CS is given. We characterize semi-simple Artinian rings in terms of CMS modules. We prove that over a right hereditary ring $R$, every projective right $R$-module is a CMS module.

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1 Introduction

Throughout this paper, by a ring $R$ we always mean an associative ring with identity and every $R$-module is unitary. Let $M$ be a right $R$-module, a right $R$-module $N$ is called $M$-cyclic if it is isomorphic to $M/L$ for some submodule $L$ of $M$. A submodule $N$ of $M$ is called an essential submodule if $N \cap B \neq 0$, for each non-zero submodule $B$ of $M$. A submodule $N$ of a right $R$-module $M$ is called closed if $N$ has no proper essential extension inside $M$. 
In 1965, Utumi [10] investigated the condition \((C_1)\). We recall that a module \(M\) satisfies the \((C_1)\) condition if every submodule of \(M\) is essential in a direct summand of \(M\). For more details on this condition, we refer [3] and [7]. Because of the disparate nature of the development of the theory, different authors have adopted different terminologies for the modules which satisfy the \((C_1)\) condition. In 1977, Chatters and Hajarnavis [1] used the term CS module. Note that a module satisfies \((C_1)\) condition if and only if every closed submodule is a direct summand of the module. In 1982, Harada and his collaborators [4] used the term extending modules as the dual to lifting modules.

Note that closed submodules and \(M\)-cyclic submodules of a module are different. Let \(R = \begin{pmatrix} F & F \oplus F \\ 0 & F \end{pmatrix}\), where \(F\) is a field and \(M_R = \begin{pmatrix} F & F \oplus F \\ 0 & 0 \end{pmatrix}\).

Then \(P_R = \begin{pmatrix} 0 & F \oplus 0 \\ 0 & 0 \end{pmatrix}\) and \(Q_R = \begin{pmatrix} 0 & 0 \oplus F \\ 0 & 0 \end{pmatrix}\) are the closed submodules of \(M_R\). But they are not \(M\)-cyclic submodules. Moreover, consider the set of integers \(Z\) as \(Z\)-module. Then every submodule of \(Z_Z\) is an \(M\)-cyclic submodule but is not a closed submodule.

The notion of \(M\)-cyclic submodules has been used in [8], [2], etc. to define some new structure of modules. The notion of these modules and CS modules motivated us to define the CMS modules by using closed submodules and \(M\)-cyclic submodules together. An example of CMS module which is not CS is given. Here, we investigate some characterization of rings by CMS (CS) modules, which have not been studied earlier. We prove that a commutative ring \(R\) is semi-simple Artinian if and only if the direct sum of any two CMS modules is a CMS module. Finally, we show that over a hereditary ring every projective module is CMS. We provide some open problems for readers.

The following implications are well known:

Injective \(\Rightarrow\) quasi-injective \(\Rightarrow\) continuous \(\Rightarrow\) quasi-continuous \(\Rightarrow\) CS.

The above implications can be extended as follows:

Injective \(\Rightarrow\) quasi-injective \(\Rightarrow\) continuous \(\Rightarrow\) quasi-continuous \(\Rightarrow\) CS \(\Rightarrow\) CMS.

2 CMS Modules

Definition 2.1 A right \(R\)-module \(M\) is called CMS if every closed \(M\)-cyclic submodule of \(M\) is a direct summand of \(M\).

The class of CMS modules is bigger than the class of CS modules. We justify it by the following example of CMS module which is not a CS module.

Example 2.2 Let \(Q\) and \(Z_p\) be two \(Z\)-modules, where \(Q\) is the set of rational numbers, \(Z\) is the ring of integers and \(p\) is any prime integer. Consider
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the $Z$-module $M = Q \oplus Z_p$. The only closed $M$-cyclic submodules of $M$ are 0, $Q$, $Z_p$ and $M$. It follows that $M$ is CMS but by Example 10 of [9], $M$ is not a CS module.

**Proposition 2.3** Any direct summand of a CMS module is a CMS module.

**Proof.** Let $N$ be any direct summand of a CMS module $M$ such that $M = N \oplus K$ for some submodule $K$ of $M$. Then $N$ is a closed $M$-cyclic submodule of $M$. Suppose $X$ is a closed $N$-cyclic submodule of $N$. Then it is obvious that $X$ is a closed $M$-cyclic submodule of $M$. Since $M$ is a CMS module therefore $X$ is a direct summand of $M$ such that $M = X \oplus L$ for some submodule $L$ of $M$. Since $M = X \oplus L = N \oplus K$ and $X \subseteq N$ therefore $X$ is a direct summand of $N$. Hence $N$ is a CMS module.

There are some modules which are not CMS. Note that the direct sum of two CMS modules need not be CMS. For example, $M_Z = Z \oplus Z_p$, where $p$ is any prime integer and $Z$ is the set of integers. Since $Z$ is uniform and $Z_p$ is a simple $Z$-module, therefore $Z$ and $Z_p$ are CMS modules. But $M_Z$ is not a CMS module because $2Z$ is a closed $M$-cyclic submodule of $M_Z$ and it is not a direct summand of $M_Z$.

If the direct sum of any two CMS modules is CMS, then in the following theorem, we characterize commutative semi-simple Artinian rings.

**Theorem 2.4** Let $R$ be a commutative ring. Then $R$ is semi-simple Artinian if and only if the direct sum of any two CMS modules is CMS module.

**Proof.** Let $M$ be a CMS module and $E(M)$ be the injective hull of $M$. Suppose the direct sum of any two CMS modules is CMS module. Then $M \oplus E(M)$ is a CMS module. Note that if $M \oplus E(M)$ is a CMS module, then any homomorphism from a closed $M \oplus E(M)$-cyclic submodule $X$ of $M \oplus E(M)$ to $M \oplus E(M)$ can be extended to an endomorphism of $M \oplus E(M)$. Therefore, we can show that $M$ is isomorphic to a direct summand of $E(M)$. Hence $M$ is injective.

Since every simple module is a CMS module, it is injective. So, $R$ is a $V$-ring and thus a von-Neumann regular because $R$ is a commutative ring. Moreover, every completely-reducible module is a CMS module, therefore it is injective. By [6], it follows that if the countable direct sum of injective hulls of simple modules is injective then $R$ is a Noetherian ring. Thus $R$ being Noetherian and regular is semi-simple Artinian. The converse is obvious.

**Corollary 2.5** Let $R$ be a commutative ring. Then $R$ is semi-simple Artinian if and only if the direct sum of any two CS modules is CS module.
Remark 2.6 Since the idea of CMS modules is a generalisation of CS modules therefore it contains alot of open problems. For example, when a CMS module can be written as direct sums of uniform submodules? and another problem is that over which ring every R-module is a CMS module? We are unable to find the full answers of these questions. In the following theorem, we provide partial answer to that.

**Proposition 2.7** Let \( M \) be a projective right \( R \)-module. Then \( M \) is a CMS module if and only if every closed \( M \)-cyclic submodule of \( M \) is projective.

**Proof.** Suppose \( M \) is a CMS module and \( X \) is a closed \( M \)-cyclic submodule of \( M \). Therefore \( X \) is a direct summand of \( M \). Since \( M \) is projective, \( X \) is also projective. Conversely, suppose that \( X \) is a projective closed \( M \)-cyclic submodule of \( M \). Therefore there is an epimorphism \( \pi : M \rightarrow X \). Let \( I : X \rightarrow X \) be the identity map. Then \( I \) can be lifted to a homomorphism \( f \) from \( X \) to \( M \). Therefore \( X \) is a direct summand of \( M \). Hence \( M \) is a CMS module.

A ring \( R \) is called right hereditary if every right ideal is projective. Moreover, \( R \) is right hereditary if and only if every submodule of any projective right \( R \)-module is projective. Note that a projective module need not be CMS. For example, \( M \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}_p \) is a free module. Therefore, \( M \) is a projective module. But \( M \) is not CMS. In the following, we show that over a right hereditary ring every projective module is CMS.

**Theorem 2.8** Let \( R \) be a right hereditary ring. Then every projective right \( R \)-module is a CMS module.

**Proof.** Suppose \( R \) is a right hereditary ring and \( M \) is a projective right \( R \)-module. Then any closed \( M \)-cyclic submodule \( X \) of \( M \) is projective. By Proposition 2.10, \( X \) is a direct summand of \( M \) and hence \( M \) is a CMS module.

We recall that a module \( M \) is with the CM-property if, every closed submodule of \( M \) is an \( M \)-cyclic submodule of \( M \). Note that CS modules satisfy the CM-property. But CMS modules need not satisfy the CM-property. We state a sufficient condition for the CMS modules to be CS.

**Proposition 2.9** A right \( R \)-module \( M \) is CS if and only if \( M \) is a CMS module and satisfies the CM-property.

**Proof.** Straightforward.
References


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