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# $\mathfrak{C}_8$-Groups and Nilpotency Condition

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**Abstract**

A cover for a group is a collection of proper subgroups whose union is the whole group. A cover is irredundant if no proper sub-collection is also a cover and is called maximal if all its members are maximal subgroups. For an integer $n > 2$, a cover with n members is called an *n-cover*. In this paper we characterize all nilpotent groups have a maximal irredundant 8-cover with core-free intersection.

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**Keywords:** covering groups by subgroups, maximal iredundant cover,nilpotent groups, core-free intersection.

## 1 Introduction and results

Let $G$ be a group. A set $\mathcal{C}$ of proper subgroups of $G$ is called a cover for $G$ if its set-theoretic union is equal to $G$. If the size of $\mathcal{C}$ is $n$, we call $\mathcal{C}$ an $n$-cover for the group $G$. A cover $\mathcal{C}$ for a group $G$ is called irredundant if no proper subset of $\mathcal{C}$ is a cover for $G$. A cover $\mathcal{C}$ for a group $G$ is called core-free if the intersection $D = \bigcap_{M\in\mathcal{C}} M$ of $\mathcal{C}$ is core-free in $G$, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of $G$. A cover $\mathcal{C}$ for a group $G$ is called maximal if all the members of $\mathcal{C}$ are maximal subgroups of $G$. A cover $\mathcal{C}$ for a group $G$ is called a $\mathfrak{C}_n$-cover whenever $\mathcal{C}$ is an irredundant maximal core-free $n$-cover for $G$ and in this case we say that $G$ is a $\mathfrak{C}_n$-group.

In [14], Scorza determined the structure of all groups having an irredundant 3-cover with core-free intersection.

**Theorem 1.1** (Scorza [14]) *Let $\{A_i\ :\ 1 \le i \le 3\}$ be an irredundant cover with core-free intersection $D$ for a group $G$. Then $D = 1$ and $G \cong C_2 \times C_2$.*

In [10], Greco characterized all groups having an irredundant 4-cover with core-free intersection.

**Theorem 1.2** (Greco [10]) *Let $\{A_i : 1 \leq i \leq 4\}$ be an irredundant cover with core-free intersection $D$ for a group $G$. If the cover is maximal then either*

1. *$D = 1$ and $G \cong Sym_3$ or $G \cong C_3 \times C_3$;*
   *or*

2. *$|D| = 2$, $|G| = 18$ and $G$ embeds into $Sym_3 \times Sym_3$.*

*If the cover is not maximal then either*

1. *$D = 1$ and $G \cong D_8$, or $G \cong C_4 \times C_2$, or $G \cong C_2 \times C_2 \times C_2$;*
   *or*

2. *$|D| = 2$ and $G \cong D_8 \times C_2$, where $D_8$ is the dihedral group of order 8.*

Bryce et al. [5] characterized groups with maximal irredundant 5-cover with core-free intersection.

We characterized groups with maximal irredundant 6-cover with core-free intersection in [5].

Abdollahi et al. [3] characterized groups with maximal irredundant 7-cover with core-free intersection.

Also we characterized $p$-groups with maximal irredundant 8-cover with core-free intersection in [2].

**Theorem 1.3** ( *See [2]* ). *Let $G$ be a $\mathfrak{C}_8$-group. Then $G$ is a $p$-group for a prime number $p$ if and only if $G \cong (C_3)^4$ or $(C_7)^2$.*

Here we characterize all nilpotent groups having a maximal irredundant 8-cover with core-free intersection.

Further problems of a similar nature, with slightly different aspects, have been studied by many people (see [1,2,4,5,7,9,14]).

We shall use the following lemma frequently and we state it here for reader convenience.

**Lemma 1.4** (*Lemma 2.2 of [5]*). *Let $\Gamma = \{A_i : 1 \leq i \leq m\}$ be an irredundant covering of a group $G$ whose intersection of the members is $D$.*

*(a) If $p$ is a prime, $x$ a $p$-element of $G$ and $|\{i : x \in A_i\}| = n$ , then either $x \in D$ or $p \leq m - n$.*

*(b) $\bigcap_{j \neq i} A_j = D$ for all $i \in \{1, 2, ..., m\}$.*

*(c) If $\bigcap_{i \in S} A_i = D$ whenever $|S| = n$, then $\left|\bigcap_{i \in T} A_i : D\right| \leq m - n + 1$ whenever $|T| = n - 1$.*

*(d) If $\Gamma$ is maximal and $U$ is an abelian minimal normal subgroup of $G$, then if $|\{i : U \subseteq A_i\}| = n$, either $U \subseteq D$ or $|U| \leq m - n$.*

# 2 Nilpotent groups with a maximal irredundant 8-cover

In this section we characterize all nilpotent $\mathfrak{C}_8$-groups.

**Theorem 2.1** *A group $G$ having a maximal irredundant 8-cover with core-free intersection is nilpotent if and only if $G \cong C_3 \times C_3 \times C_3 \times C_3$ or $G \cong C_7 \times C_7$.*

**proof** Suppose that $G$ is a nilpotent group. Firstly by the hypothesis we have that $M_i$ is normal in $G$ for all $i \in \{1,2,3,4,5,6,7,8\}$ and $D = 1$. Thus every prime divisor $p$ of $|G|$ is at most seven. Hence $G$ is a $\{2,3,5,7\}$-group and by the hypothesis

$$G \cong (C_2)^i \times (C_3)^j \times (C_5)^k \times (C_7)^l \text{ that } i+j+k+l \leq 7.$$

By Lemma 1.4, every non-trivial 7-element lies in exactly one $M_i$. Assume that 7 divides $|G|$. Then there is at most one $M_i$ whose index is either 2 or 3 or 5. Thus there are at least seven $M_i$s of index 7. Since $\bigcap_{i \in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$, we have that $G \cong (C_7)^l$. Now Theorem 1.3 implies that $G \cong C_7 \times C_7$.
Therefore we may assume that $G \cong (C_2)^i \times (C_3)^j \times (C_5)^k$.

Suppose, for a contradiction, that $j = 0$ and $i, k \geq 1$, therefore $G \cong (C_2)^i \times (C_5)^k$. Now by Lemma 1.4, every non-trivial 5-element of $G$ lies in at most three $M_i$s, therefore at most three of $M_i$s have indices two. Thus we consider the following cases:

**Case 1** If the 5-elements of $G$ lie in exactly one $M_i$, then at most one of the $M_i$, has index two.

So seven of $M_i$s have indices five. Since $\bigcap_{i \in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$, thus $G \cong (C_5)^k$ which is a contradiction, since $i, k \geq 1$.

**Case 2** If the 5-elements of $G$ lie in exactly two $M_i$s, then only two $M_i$s have indices two. Thus 4 divides $|G|$. Also exactly six $M_i$s have indices five. Now since $\bigcap_{i \in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$, thus $|G| \leq 2 \times 5^6$. So for some positive integer $m$ with $m \leq 6$, we have $|G| = 2 \times 5^m$ which is a contradiction, since 4 divides $|G|$.

Thus we conclude that all of 5-elements of $G$ lie in exactly three $M_i$s, therefore three of $M_i$s, have indices two, say $M_1$, $M_2$ and $M_3$, and $|G : M_i| = 2$, $i \in \{1,2,3\}$.

Suppose that $|G : M_j| = 5$ for $j \in \{4,5,6,7,8\}$. Hence all of 5-elements of $G$ lie in $M_1 \cap M_2 \cap M_3$. Therefore, by Lemma 1.4 (a) $M_\ell$ contains no non-trivial 5-element for every $\ell \in \{4,5,6,7,8\}$. Since $G$ is abelian, it has a unique Sylow 2-subgroup. It follows that $M_4 = M_5 = M_6 = M_7 = M_8$, a contradiction. Hence $G \ncong (C_2)^i \times (C_5)^k$.

Now if $G \cong (C_2)^i \times (C_3)^j$ where $i, j \geq 2$, then there are at most five of $M_i$s that have indices two. Also since every nontrivial 2-element of $G$ lie in at most six $M_i$s, therefore there are at most six $M_i$s, whose indices are three. There are at least two $M_i$s, whose indices are two. Thus we consider the following cases:

**Case 1** If there are exactly two $M_i$s, whose indices are two, say $M_1$ and $M_2$, then 4 divides $|G|$. Since $\bigcap_{i\in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$, we have that $|G : \bigcap_{i=2}^{8} M_i| \leq 2 \times 3^6$. So for some positive integer $m$ with $m \leq 6$, we have $|G| = 2 \times 3^m$ which is a contradiction, since 4 divides $|G|$.

**Case 2** If there are exactly three $M_i$s, whose indices are two, say $M_1$, $M_2$ and $M_3$, then $2^3$ divides $|G|$, since $|G : M_1 \cap M_2 \cap M_3| \leq 2^3$. We have $|G : \bigcap_{i=2}^{8} M_i| \leq 2^2 \times 3^5$, since $\cap_{i\in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$. Thus $|G| = 2^2 \times 3^m$ for some positive integer $m$ with $m \leq 5$, which is a contradiction, since 8 divides $|G|$.

**Case 3** If there are exactly four $M_i$s, whose indices are two, say $M_1$, $M_2$, $M_3$ and $M_4$. Then $2^4$ divides $|G|$, since $|G : M_1 \cap M_2 \cap M_3 \cap M_4| \leq 2^4$. We have $|G : \bigcap_{i=2}^{8} M_i| \leq 2^3 \times 3^4$, since $\bigcap_{i\in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$. Thus $|G| \leq 2^3 \times 3^4$, a contradiction, since $2^4$ divides $|G|$.

Thus there are exactly five $M_i$s, whose indices are two, say $M_1$, $M_2$, $M_3$, $M_4$ and $M_5$. Therefore all 3-elements of $G$ lie in $M_1 \cap M_2 \cap M_3 \cap M_4 \cap M_5$. Therefore, by Lemma 1.4 (a) $M_\ell$ contains no non-trivial 3-element for every $\ell \in \{6,7,8\}$. Since $G$ is abelian, it has a unique Sylow 2-subgroup. It follows that $M_6 = M_7 = M_8$, a contradiction. Hence $G \ncong (C_2)^i \times (C_3)^j$.

Now we let $G \cong (C_3)^j \times (C_5)^k$ where $j, k \geq 2$, so by Lemma 1.4 (a) every 5-element of $G$ lies in at most three of $M_i$s, therefore at most three of $M_i$s have indices three. Therefore we consider the following cases:

**Case (i)** If the 5-elements are in only one $M_i$, then at most one of the $M_i$ has index three. So at least seven of $M_i$s have indices five. Since $\bigcap_{i\in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$, therefore $G \cong (C_5)^k$, a contradiction.

**Case (ii)** If the 5-elements lie in exactly two $M_i$s, then only two of the $M_i$s have indices three, thus 9 divides $|G|$. Also exactly six of $M_i$s have indices five. Since $\bigcap_{i\in T} M_i = 1$ for all $T \subset \{1,2,3,4,5,6,7,8\}$ with $|T| = 7$, we have $|G| \leq 3 \times 5^6$. So for some positive integer $m$ with $m \leq 6$, we have $|G| = 3 \times 5^m$. Which is a contradiction, since 9 divides $|G|$.

Thus we conclude that all of 5-elements of $G$ lie in exactly three $M_i$s, therefore three of the $M_i$, have indices three, say $M_1$, $M_2$ and $M_3$, with $|G : M_i| = 3$, $i \in \{1,2,3\}$.

Suppose that $|G : M_j| = 5$ for $j \in \{4,5,6,7,8\}$. Hence all of 5-elements of $G$ lie in $M_1 \cap M_2 \cap M_3$. Therefore, by Lemma 1.4 (a) $M_\ell$ contains no non-trivial 5-element for every $\ell \in \{4,5,6,7,8\}$. Since $G$ is abelian, it has a unique Sylow

3-subgroup. It follows that $M_4 = M_5 = M_6 = M_7 = M_8$, a contradiction. Hence $G \ncong (C_3)^j \times (C_5)^k$.

Now let $G \cong (C_2)^i \times (C_3)^j \times (C_5)^k$ where $i, k \ngeq 0$, then $i + j + k \leq 7$. Let $k \neq 0 \neq i$. By Lemma 1.4 (a) every 5-element of $G$ lies in at most three of $M_i$s. Therefore at most three of $M_i$s have indices two or three.

Thus at least five $M_i$s, have indices five, and since $\bigcap_{i\in T} M_i = 1$ for all $T \subset \{1, 2, 3, 4, 5, 6, 7, 8\}$ with $|T| = 7$, we have that $|G : \bigcap_{i=2}^{8} M_i| \leq 3^2 \times 5^5$. So for some positive integers $m$ and $l$ with $m + l \leq 7$, we have $|G| = 3^l \times 5^m$ which is the final contradiction. Therefore $G \cong (C_3)^j$ and by Theorem 1.3 we have $G \cong C_3 \times C_3 \times C_3 \times C_3$.□

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