On V-regular Semigroups

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Abstract

A regular semigroup $S$ is V-regular if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S$. A characterization of a V-regular semigroup is given. Congruences on V-regular semigroups are described in terms of certain congruence pairs.

Mathematics Subject Classification: 20M17

Keywords: regular semigroup, V-regular semigroup, congruence, congruence pair

1 Introduction and Preliminaries

A regular semigroup $S$ is called $V$-regular if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S$. This concept was introduced by Onstad [8]. This class of semigroups is dual to orthodox semigroups, namely, regular semigroups satisfy that $V(b)V(a) \subseteq V(ab)$ for all elements $a, b$ in the semigroup. Properties of V-regular semigroups were given by Nambooripad and Pastijn in [7].

Congruences on regular semigroups have been explored extensively. The kernel-trace approach is an effective tool for handling congruences on regular semigroups, which had been investigated in the previous literature, such as Crvenković and Dolinka [1], Feigenbaum [2], Gomes [3, 4], Imaoka [6], Pastijn
and Petrich [9], Petrich [10], Scheiblich [11], Trotter [12, 13] and the author [14].

The purpose of this paper is to give a characterization of a V-regular semigroup, and to describe congruences on V-regular semigroups in terms of certain congruence pairs.

For standard terminology and notation in semigroup theory see Howie [5].

If $S$ is a regular semigroup, $a \in S$, then $V(a)$ denotes the set of inverses of $a$ in $S$. The set of idempotents of $S$ is denoted by $E(S)$. On $E(S)$ we shall consider the natural partial order $\omega$ given by

$$e \omega f \iff ef = fe = e.$$ 

For $e, f \in E(S)$, $S(e, f) = fV(ef)e$ is the sandwich set of $e$ and $f$.

The following simple statements will be applied without further mention: for $e, f \in E(S)$,

$$eLf \Rightarrow S(e, f) = \{f\},$$

$$eRf \Rightarrow S(e, f) = \{e\}.$$ 

If $\rho$ is a congruence on $S$ and $h \in S(e, f)$, then $h\rho \in S(e\rho, f\rho)$.

Let $\tau$ be a relation on $S$. The congruence generated by $\tau$ is denoted by $\tau^*$.

If $\gamma$ is an equivalence on $S$, then $\gamma^0$ is the greatest congruence on $S$ contained in $\gamma$. $C(S)$ is the lattice of congruences on $S$.

**Lemma 1.1.** [7] A regular semigroup $S$ is V-regular if and only if the partial band $(E(S), \circ)$ determined by $S$ satisfies the following:

1. $\omega L = L \omega$;
2. $\omega R = R \omega$;
3. for all $e, f \in E(S), h \in S(e, f)$ there exist $e_1, f_2 \in E(S)$ such that $e_1Le, f_2Rf$, and $h = f_2e_1$.

**Lemma 1.2.** [5] Let $S$ be a regular semigroup, $\rho \in C(S)$. If $a\rho \in E(S/\rho)$, then there exists $e \in E(S)$ such that $a\rho = e\rho$.

**Lemma 1.3.** Let $S$ be a V-regular semigroup, $\rho \in C(S), a\rho \in E(S/\rho), x\rho \in S/\rho$. If $(a\rho)R(x\rho)$ in $S/\rho$, then there exists $e \in E(S)$ such that $a\rho = e\rho$ and $eRx$.

**Proof.** By Lemma 1.2, there exists $f \in E(S)$ such that $a\rho = f\rho$. Let $g \in E(S)$ be such that $gRx$. Then $(g\rho)R(x\rho)$. Since $a\rho = f\rho$ and $(a\rho)R(x\rho)$, we have $(f\rho)R(g\rho)$. Let $h \in S(f, g)$. Then $h\rho \in S(f\rho, g\rho)$, and so $h\rho = f\rho$. Notice that $hg \in E(S), hR(hg)\omega g$, it follows from Lemma 1.1 that there exists $e \in E(S)$ such that $h\omega e\rho$. Since $gRx, eRx$. Now $(h\rho)\omega(e\rho)R(g\rho)$ implies that $(f\rho)\omega(e\rho)R(f\rho)$. Hence $a\rho = f\rho = f\rho \cdot e\rho = e\rho$.  \[\square\]
Corollary 1.4 Let $S$ be a $V$-regular semigroup, $\rho \in C(S), e, f \in E(S)$. If $(e\rho)\mathcal{R}(f\rho)$, then there exist $g, h \in E(S)$ such that $g\mathcal{R}f, h\mathcal{R}e, g\rho = e\rho$ and $h\rho = f\rho$.

Remark The dual results of Lemma 1.3 and Corollary 1.4 hold.

2 Main Results

The theorem below give a characterization of a $V$-regular semigroup.

Theorem 2.1. A regular semigroup $S$ is $V$-regular if and only if for all $a, b \in S, (ab)' \in V(ab)$ there exist $e_1, e_2, f_1, f_2 \in E(S)$ such that $b(ab)'a = f_2e_1, e_1LaRe_2, f_1LbRf_2, ab(ab)'\omega = 2$ and $(ab)'ab\omega f_1$.

Proof. $\Rightarrow$. Since $S$ is $V$-regular, for all $a, b \in S, (ab)' \in V(ab)$ there exist $a' \in V(a), b' \in V(b)$ such that $(ab)' = b'a'$. Let

$$e_1 = a'a, f_1 = b'b, e_2 = aa', f_2 = bb'.$$

Then $e_1, e_2, f_1, f_2 \in E(S)$ and

$$b(ab)'a = bb'a'a = f_2e_1, e_1 = a'aLaRa = e_2, f_1 = b'bLbRb = f_2.$$

Now

$$(ab)(ab)'e_2 = (ab)(ab)'aa = (ab)(b'aa') = (ab)b'a' = (ab)(ab)'$$

and

$$e_2(ab)(ab)' = (aa')(ab)(ab)' = (aa'a)b(ab)' = (ab)(ab)' .$$

It follows that $(ab)(ab)' \omega e_2$.

Similarly, $(ab)'ab\omega f_1$.

$\Leftarrow$. Let $a, b$ satisfy the condition stated in the theorem. Now $e_1LaRe_2, f_1LbRf_2$ imply that there exist

$$a' \in V(a) \cap (Le \cap Re_1), b' \in V(b) \cap (Lf \cap Rf_1)$$

such that

$$a'a = e_1, aa' = e_2, b'b = f_1, bb' = f_2.$$

Since $b(ab)'a = f_2e_1$, we have that

$$b'a' = (b'f_2)(e_1a') = b'(f_2e_1)a' = b'(ab)'aa'.$$

Thus

$$(b'a')(ab)(b'a') = (b'b(ab)'aa')(ab)(b'b(ab)'aa')$$

$$= b'b(ab)'ab(ab)'aa'$$

$$= b'b(ab)'aa' = b'a'$$
and 
\[(ab)(b'a')(ab) = ab(b'ab')ab = ab(ab)'ab = ab,\] 
that is, \(b'a' \in V(ab).\) 

Also 
\[(b'a')(ab) = (b'b(ab')ab = b'b(ab)'ab \]
\[= f_1(ab)'ab \]
\[= (ab)'ab \quad (\text{since } (ab)ab \omega f_1)\] 

and 
\[(ab)(b'a') = (ab)(b'b(ab')ab = ab(ab)'ab' \]
\[= (ab)(ab)e_2 \]
\[= (ab)(ab)' \quad (\text{since } (ab)(ab)' \omega e_2).\] 

It follows that 
\[(ab)' = (ab)'(ab)(ab)' = (b'a')(ab)(ab)' = (b'a')(ab)(b'a') = b'a'.\] 

Therefore, \(S\) is V-regular. \(\square\)

**Theorem 2.2.** Let \(S\) be a V-regular semigroup, \(\rho \in C(S), a, b \in S.\) If \(a \rho b,\) 
then for any \(a' \in V(a)\) there exists \(b' \in V(b)\) such that \(a' \rho b'.\)

**Proof.** Let \(a' \in V(a).\) Then \(a' \rho \in V(ap),\) Since \(a \rho b,\) we have that \(a' \rho \in V(ap) = V(bp).\) Let \(f \rho = bp \cdot a' \rho, f' \rho = a' \rho \cdot bp.\) Then 
\[(f \rho)R(bp), (f' \rho)L(bp), f \rho, f' \rho \in E(S/\rho).\]

By Lemma 1.3 and its dual, there exist \(e, e' \in E(S)\) such that \(e R b L e', f \rho = e \rho\) and \(f' \rho = e' \rho.\)

Take \(b' \in V(b) \cap L_e \cap R_{e'}.\) Then \(b' \rho \in L_{e \rho} \cap R_{e' \rho}.\) Hence 
\[b' \rho = e' \rho \cdot b' \rho \cdot e \rho = f' \rho \cdot b' \rho \cdot f \rho = a' \rho \rho b \cdot b' \rho \cdot b \rho a' \rho \]
\[= a' \rho \cdot b \rho b' \rho b \rho \cdot a' \rho = a' \rho \cdot b \rho \cdot a' \rho = a' \rho \cdot a \rho \cdot a' \rho = a' \rho,\]

that is, \(a' \rho b'.\) \(\square\)

To provide a characterization of congruences on V-regular semigroups in terms of certain congruence pairs, we need the following results.

**Lemma 2.3.** Let \(S\) be a V-regular semigroup, \(\rho \in C(S)\) with \(\tau = tr \rho.\)

(1) \((e \rho)R(f \rho)\) in \(S/\rho \Leftrightarrow e(\tau R) f \) in \(S \Leftrightarrow e(\tau R)f \) in \(S \); (e, f \( \in E(S)).\))
(2) \(R \tau R = R \tau R.\)

**Proof.** (1) Let \(e, f \in E(S)\) be such that \((e \rho)R(f \rho)\) in \(S/\rho.\) By Corollary 1.4, there exist \(g, h \in E(S)\) such that 
\[gRf, hRg = e \rho, h \rho = f \rho.\]
Thus \( e\rho gRf, eRh\rho f, \) whence \( e(\tau R)f, e(\tau R)f. \)

If conversely \( e(\tau R)f, \) then exists \( g \in E(S) \) such that \( e\tau gRf, \) and so \( (e\rho) = (g\rho)R(f\rho). \)

Similarly, \( e(\tau R)f \) implies that \( (e\rho)R(f\rho). \)

(2) Obviously, \( R\tau R\tau R \supseteq R\tau R. \)

If \( a(\tau R\tau R)b \) for \( a, b \in S, \) then by [9, Lemma 2.6 (ii)] we have \( (a\rho)R(bp) \) in \( S/\rho. \) Hence for \( a' \in V(a), b' \in V(b), \) we have

\[
(a'\rho)R(a\rho)R(bp)R(bb'\rho).
\]

Since \( aa', bb' \in E(S), \) by part (1) we have \( (aa')\tau R(bb') \) and thus

\[
aR(aa')\tau R(bb')Rb,
\]

whence \( a(\tau R)b. \)

An equivalence \( \tau \) on the set \( E(S) \) of idempotents of a regular semigroup \( S \) is normal if \( \tau = \text{tr} \tau^* \) [9]. It follows from Lemma 2.3 [9] that an equivalence \( \tau \) on \( E(S) \) is normal if and only if \( \tau \) is the trace of a congruence on \( S. \)

Let \( K \) be a subset of a regular semigroup \( S. \) A congruence \( \rho \) on \( S \) saturates \( K \) if \( a \in K \) implies \( a\rho \subseteq K. \) The greatest congruence on \( S \) which saturates \( K \) is denoted by \( \pi_K. \) Recall from Result 1.5 [9] that for \( a, b \in S, a\pi_K b \) if and only if

\[
 xay \in K \iff xby \in K \ (x, y \in S^1),
\]

and \( \pi_K = \theta_K^0, \) where the equivalence relation \( \theta_K \) on \( S \) is defined by

\[
a\theta_K b \iff a, b \in K \text{ or } a, b \in S \setminus K.
\]

A subset \( K \) of a regular semigroup \( S \) is normal if \( K = \ker \pi_K \) [9]. Recall from [9] that a subset \( K \) of \( S \) is normal if and only if \( K \) is the kernel of a congruence on \( S. \)

The pair \( (K, \tau) \) is a congruence pair for a regular semigroup \( S \) (see [9]) if

(i) \( K \) is a normal subset of \( S, \)
(ii) \( \tau \) is a normal equivalence on \( E(S), \)
(iii) \( K \subseteq \ker (L\tau L\tau L \cap R\tau R\tau R)^0, \)
(iv) \( \tau \subseteq \text{tr} \pi_K. \)

In such a case \( \rho(K, \tau) \) is defined by

\[
\rho(K, \tau) = \pi_K \cap (L\tau L\tau L \cap R\tau R\tau R)^0.
\]

Note that

\[
\rho(K, \tau) = (L\tau L\tau L \cap \theta_K \cap R\tau R\tau R)^0.
\]

When \( S \) is a V-regular semigroup it follows from Lemma 2.3 (2) and its dual result that

\[
\rho(K, \tau) = (L\tau L \cap \theta_K \cap R\tau R)^0.
\]
The characterization of congruences on a V-regular semigroup in terms of congruence pairs follows from [9, Theorem 2.13].

**Theorem 2.4.** If \((K, \tau)\) is a congruence pair for a V-regular semigroup \(S\), then \(\rho_{(K,\tau)}\) is the unique congruence on \(S\) such that \(\ker \rho_{(K,\tau)} = K\) and \(\text{tr} \rho_{(K,\tau)} = \tau\). Conversely, if \(\rho\) is a congruence on \(S\), then \((\ker \rho, \text{tr} \rho)\) is a congruence pair for \(S\) and \(\rho = \rho_{(\ker \rho, \text{tr} \rho)}\).

**References**


**Received: April, 2010**