

A Note on the Negative Pell Equation

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Abstract

We provide a new elementary proof of a criterion, given in earlier work, for the solvability of the Pell equation $x^2 - Dy^2 = -1$ where D is any positive nonsquare integer.

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1 Introduction

The Pell equation is perhaps the oldest diophantine equation that has interested mathematicians all over the world for probably more than a 1000 years now. The literature is full of articles about this equation, a recent excellent reference and overview being [2], where the reader will find historical and up-to-date references. Henceforth n will denote a positive integer that is not a square.

It is well known that the positive Pell equation,

$$x^2 - ny^2 = 1 \tag{1.1}$$

has infinitely many solutions (x, y) whereas the negative Pell equation,

$$x^2 - ny^2 = -1, \tag{1.2}$$

does not always have a solution. It is also known that (1.1) has a *fundamental solution* (x_0, y_0) meaning *the least positive solution* $(x, y) = (x_0, y_0)$ namely $x_0 > 0$ and $y_0 > 0$ have least value. This is well defined since it may be shown that if $a + b\sqrt{D}$ and $e + f\sqrt{D}$ are both positive solutions of $x^2 - Dy^2 = c$, for any nonzero $c \in \mathbb{Z}$, then the following are equivalent: (1) $a < e$, (2) $b < f$, (3) $a + b\sqrt{D} < e + f\sqrt{D}$ —see [6] for verification and a generalization of this to solutions of the Diophantine equation $d_1x^2 - d_2y^2 = \pm 1$. It is well known that all solutions (x, y) to (1.1) are given by

$$x + y\sqrt{n} = (x_0 + y_0\sqrt{n})^k,$$

where k is any integer. Similarly equation (1.2) also has a fundamental solution (a_0, b_0) where now all solutions are given by odd powers k of (a_0, b_0) . Moreover if (1.2) is solvable with fundamental solution (a_0, b_0) , then the fundamental solution of (1.1) is given by

$$x_0 + y_0\sqrt{n} = (a_0 + b_0\sqrt{n})^2. \quad (1.3)$$

These two problems, namely, the determination of a non-trivial solution of (1.1) efficiently and deciding, in a short time, whether (1.2) has a solution are centuries-old problems. While it is known that (1.1) always has a non-trivial solution, there is no easy way to find this solution. Indeed the continued fraction expansion of \sqrt{n} does give the fundamental solution, however this method is not efficient for large n as its running time is $O(\sqrt{n})$. Indeed even for small n , sometimes the solutions can be so large that even writing the solution down may be time consuming (in the sense that the number of digits in x or y is too big.) For example for $n = 217$ we have $(x, y) = (3844063, 260952)$ is a solution to (1.1). On the other hand large values of n can have small solutions, for example for $n = 10353$ we have $(x, y) = (407, 4)$ as a solution of the positive Pell equation. There does not appear to be a design as to why this occurs. In an excellent paper [5], Lenstra presents an algorithm based on smooth numbers and a concept of power products that finds, rather represents, a solution to (1.1) in running time that is conjecturally faster than the traditional method using continued fractions.

Turning to the negative Pell equation (1.2), we have already remarked that it does not always have a solution. Therefore the problem in this case is to determine when it is solvable. Again the continued fraction method will answer this question easily, however it is not an efficient method in terms of the time taken. Lagarias in [4] presents an algorithm, which while faster than the continued fraction method still runs in time that is exponential in n . However, he also presents an algorithm, that given a complete factorization of n decides whether or not (1.2) has a solution in a short time. Unfortunately this algorithm assumes the truth of the generalized Riemann hypothesis. Moreover,

as factorization is a hard problem by itself, it is clear that we do not yet have an easy way to decide the solvability of (1.2).

Given the complexity of determining whether or not (1.2) is solvable, a natural quest would be for criteria under which (1.2) is solvable. There do exist criteria such as [3], [1] in the literature, but none is simple yet to yield an efficient algorithm. In [7] we proved the following criterion which followed as a corollary to a theorem on central norms, that is of independent interest with applications to general Pell equations of the form $x^2 - ny^2 = c$. Here we give a short and completely elementary proof of this criterion.

Theorem 1.1 If $n \equiv 1, 2 \pmod{4}$ is a non-square integer, then there is a solution to $x^2 - ny^2 = -1$ if and only if $x_0 \equiv -1 \pmod{2n}$, where (x_0, y_0) is the fundamental solution of $x^2 - ny^2 = 1$.

Proof. If (1.2) is solvable with fundamental solution (a_0, b_0) , then from (1.3) we have that $x_0 = a_0^2 + nb_0^2 = -1 + 2nb_0^2 \equiv -1 \pmod{2n}$, where (x_0, y_0) is the fundamental solution (x_0, y_0) of (1.1).

Conversely assume that the fundamental solution (x_0, y_0) of (1.1) satisfies $x_0 \equiv -1 \pmod{2n}$. It follows that $x_0 = -1 + 2nk$. From (1.1) we have $(-1 + 2nk)^2 - ny_0^2 = 1$, which gives

$$nk^2 - k - y_1^2 = 0,$$

where $y_0 = 2y_1$. Therefore

$$k(nk - 1) = y_1^2,$$

from which it follows that $k = r^2$ and $nk - 1 = s^2$ as $\gcd(k, nk - 1) = 1$. Thus $nk - 1 = nr^2 - 1 = s^2$ which gives $s^2 - nr^2 = -1$ and hence the negative Pell equation is solvable. \square

It is worthwhile to observe that the criterion in Theorem 1.1 serves as a bridge between the two problems mentioned herein, namely, the solvability of (1.2) and the determination of a solution of (1.1). This is because if one has a fundamental solution (x_0, y_0) of (1.1) then one merely needs to verify whether $x_0 \equiv -1 \pmod{2n}$ to decide whether or not (1.2) is solvable.

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