Modular and Classic Ideals in Directed below Join Semilattice

Jonnalagadda Venkateswara Rao¹ and Emani Sree Rama Ravi Kumar²

1) Professor of Mathematics, Mekelle University, Mekelle, Ethiopia
   venkatjonnalagadda@yahoo.co.in

2) Lecturer, Dept. of Mathematics, V.R. Siddhartha Engineering College, Vijayawada, Andhra Pradesh -520 007. India
   srrkemani@yahoo.co.in

Abstract. A Join Semilattice S is called directed below if for each a, b ∈ S there exists d ∈ S such that d ≤ a, d ≤ b. In this paper we introduce the notion of classic elements in a directed below Join semilattice and incorporated several characterizations of these elements. We establish that an element is a classic element if and only if it is both modular and distributive. We also acquire an account of a number of characterizations of modular and classic ideals of directed below join semilattices.

Keywords: Directed below semilattice, Modular element, Distributive element, Standard element.

§0 Introduction

In lattices, Standard elements and ideals have been studied extensively by [2]. Ramana Murthy [5] have studied them in join semilattices. Many results of [5] have been extended by Talukder and Noor in [6,7]. In [6,7] the authors have also introduced the notion of modular elements in joined semilattices. On the other hand [3] have studied the modular elements very briefly in a general lattice.

In this paper we give a notion of classic elements of a join semilattice directed below. We also incorporate some characterizations of modular and classic ideals of a Join semilattice.

§1 Preliminaries

1.1 Directed below semilattice: A semilattice S is called a directed below
semilattice if for all \(a, b \in S\) there exists \(d \in S\) such that \(d \leq a, d \leq b\).

### 1.2 Modular Semilattice

A join semilattice \(S\) is called modular semilattice, if for all \(x, y, z \in S\) with \(z \leq x \leq y \vee z\), there exists \(y_1\) in \(S\) with \(y_1 \leq y\) such that \(x = y_1 \vee z\).

### 1.3 Distributive Semilattice

A join semilattice \(S\) is called Distributive semilattice, if for all \(x, y, z \in S\) with \(x \leq y \vee z\) there exists \(y_1\) and \(z_1\), in \(S\) with \(y_1 \leq y\) and \(z_1 \leq z\) such that \(x = y_1 \vee z_1\).

### 1.4 Classic Element

An element \(s\) of a semilattice \(S\) is called a classic element if \(x \leq s \vee t\) for \(x, t \in S\) implies \(x = s_1 \vee t_1\) for some \(s_1, t_1\) in \(S\) with \(s_1 \leq s\) and \(t_1 \leq t\).

### 1.5 Modular Element

An element \(m\) of semilattice \(S\) is called modular element if \(x \leq m \vee y\) with \(y \leq x\) for \(x, y \in S\) in \(S\) implies the existence of \(m_1 \leq m\) such that \(x = m_1 \vee y\).

### 1.6 Classic ideal

An ideal \(T\) of semilattice \(S\) is an classic ideal if and only if \(x \leq t \vee a\) for some \(t \in T\) and \(a, \) are elements of \(S\), implies there exists \(t_1\) in \(T\) and \(a_1 \leq a\) such that \(x = t_1 \vee a_1\).

### 1.7 Modular ideal

An ideal \(M\) of a directed below Join semilattice is said to be modular if and only if \(x \leq m \vee y\) with \(y \leq x\) for some \(m \in M\) for \(x, y \in S\) implies that there exists \(m_1 \leq m\) such that \(x = m_1 \vee y\).

### 1.8 Distributive Element

An element \(d\) of a Join semilattice \(S\) is called classic distributive if \(x \leq d \vee a\), \(x \leq d \vee b\) for \(x, a, b \in S\), implies there exists \(c \in S\) such that \(c \leq a, c \leq b\) and \(x \leq d \vee c\).

### 1.9 Note

By [6],

(i) An element \(m\) of a lattice \(L\) is modular if for all \(x, y \in L\) with \(y \leq x\),

\[
x \wedge (m \vee y) = (x \wedge m) \vee y.
\]

(ii) An element \(m\) of a join semilattice \(S\) is a modular element if \(y \leq m \vee x\) with \(y \leq x\) for \(x, y \in S\), which implies the existence of \(m_1 \leq m\) such that \(y = m_1 \vee x\).

It is very easy to see that the above two definition are equivalent in the case of a lattice.

First remark that if \(a\) and \(b\) are two modular elements of a Join semilattice directed below, then \(a \vee b\) and \(a \wedge b\) (if exists) are not necessarily modular.

In the lattice \(S_1\) and \(S_2\) of Figure 1; a routine work shows that \(a\) and \(b\) are modular, but neither \(a \wedge b\) in \(S_1\) nor \(a \vee b\) in \(S_2\) is modular.
Clearly every element of a modular semilattice is modular. More over, if every element of a semilattice is modular then the semilattice is modular.

§ 2 Classic elements and Classic ideals

In this section we characterize the Classic element and Classic ideal in Joinsemilattice directed below.

2.1 Theorem: If s is a classic element and m is a modular element of a Join semilattice directed below S, then s \lor m is modular in S.

Proof: Let s be a classic element and m be a modular element of a Join semilattice directed below S. Let y \leq x \leq (s \lor m) \lor y (x, y \in S). Then x \leq s \lor (m \lor y). Since s is a classic element, there exists s\_1 \leq s and t \leq m \lor y such that x = s\_1 \lor t.

This implies x = x \lor y = s\_1 \lor t \lor y. Now, t \leq m \lor y implies t \lor y \leq m \lor y.

\Rightarrow y \leq t \lor y, we have t \lor y = m\_1 \lor y for m\_1 \leq m. Thus x = s\_1 \lor t \lor y = s\_1 \lor m\_1 \lor y, where s\_1 \lor m\_1 \leq s \lor m. Therefore, s \lor m is modular.

2.2 Theorem: An ideal M of a directed below Join semilattice S is modular if and only if x \leq m \lor y with y \leq x, for some m \in M (x, y \in S), then there exists m\_1 \in M such that x = m\_1 \lor y.

Proof: Suppose an ideal M of a semilattice (S, \lor) be a modular ideal and let x \leq m \lor y with y \leq x for x, y \in S, for some m \in M.

Then to show that there exists m\_1 in M such that x = m\_1 \lor y.

Let x \in (x \lor ((M \land (y))) = ((x \lor M) \land (y)) (since M is modular).

This implies x \leq m\_1 \lor y for some m\_1 \leq x, m\_1 \in M.

Since m\_1 \leq x and y \leq x we have m\_1 \lor y \leq x.

Therefore x = m\_1 \lor y.

Hence, there exists m\_1 in M such that x = m\_1 \lor y.

Conversely, let the given condition holds.

Suppose x \in I \lor (M \land J) with J \subseteq I, where I, J are ideals of S.

Then x \in I (or) x \leq m \lor j for some m \in M, j \in J.

Then x \lor j \leq m \lor j.

Since j \leq x \lor j and m \in M by the given condition x \lor j = m\_1 \lor j for some

m\_1 \in M.

Now x \in I and j \in J \subseteq I implies x \lor j \in I, as I is an ideal.

Also m\_1 \leq x \lor j implies m\_1 \in I.

Thus m\_1 \in I \lor M for m\_1 \leq m.

Therefore x \lor j \in (I \lor M) \land J and x \in (I \lor M) \land J.

Hence I \lor (M \land J) \subseteq (I \lor M) \land J------------- (i)

Similarly we can prove that (I \lor M) \land J \subseteq I \lor (M \land J) --------------- (ii)

Therefore from (i) & (2) (I \lor M) \land J = I \lor (M \land J).

Therefore M is modular ideal.

Now, we give some properties of classic elements.

2.3 Theorem: Every classic element of directed below Join semilattice S is a distributive element of S.

Proof: Let s be a classic element of a semilattice S. Suppose x \leq s \lor a, x \leq s \lor b for
x, a, b ∈ S. Then there exists s1 ≤ s and a1 ≤ a such that x = s1 ∨ a1 for some s1, a1 ∈ S.
Now a1 ≤ x ≤ s ∨ b. This implies a1 = s2 ∨ b1 (b1, s2 ∈ S) for s2 ≤ s, b1 ≤ b. Therefore x = s1 ∨ a1 = s1 ∨ s2 ∨ b1 ≤ s ∨ b1 where b1 ≤ a1 ≤ a, and b1 ≤ b. Therefore there exists b1 in S such that b1 ≤ a, b1 ≤ b and x ≤ s ∨ b1. Therefore s is a distributive element of S.

2.4 Theorem: Let s ∈ S be a classic element. If a ∧ s exists for some a ∈ S, then a ∧ s is a classic element in (a).

Proof : Let s ∈ S be a classic element. Suppose a ∧ s exists for some a ∈ S.
Let s ∈ (a] and x ∈ (a] and x ≤ (a ∧ s) ∨ t for t ∈ (a]. Since s is a classic element, for x ≤ s ∨ t, we have x = s1 ∨ t1 for some s1, t1 ∈ S such that s1 ≤ s and t1 ≤ t.
Then s1 ≤ s ∨ t1 = x. Then s1 ≤ x ≤ a as x ∈ (a]. Therefore s1 ≤ a ∧ s.
Therefore for x ≤ (a ∧ s) ∨ t, we have x = s1 ∨ t1 for s1 ≤ a ∧ s, t1 ≤ t.
Hence, a ∧ s is a classic element in (a).

2.5 Theorem: Let s1, s2 ∈ S be a classic elements, then s1 ∨ s2 is also a classic element.

Proof : Let s1, s2 ∈ S be two classic elements.
To prove that s1 ∨ s2 is a classic element.
Let x ≤ (s1 ∨ s2) ∨ t = s1 ∨ (s2 ∨ t)
Since s1 is a classic element, there exists a ≤ s1 and b ≤ s2 ∨ t such that x = a ∨ b.
Now b ≤ s2 ∨ t, since s2 is a classic element, there exists c ≤ s2, d ≤ t such that b = cvd. So, x = a ∨ b = a ∨ c ∨ d, where a ≤ s1, c ≤ s2 and d ≤ t.
Then a ∨ c ≤ s1 ∨ s2, d ≤ t.
Therefore for x ≤ (s1 ∨ s2) ∨ t, there exists a ∨ c and d with a ∨ c ≤ s1 ∨ s2, d ≤ t such that x = (a ∨ c) ∨ d.
Therefore s1 ∨ s2 is a classic element.

2.6 Note: An ideal I of a semilattice S is a classic ideal, then we have
X ∨ (I ∧ Y) = (X ∨ I) ∧ (X ∨ Y) for X, Y ∈ I(S), where I(S) is the set of all ideals of S.

2.7 Theorem: An element s of a semilattice S is a classic element if and only if (s] is a classic ideal of S.

Proof : Suppose s is a classic element of S. Now (a] ∨ (((s] ∧ [t]) = (a] ∨ (s ∨ t])
= (a ∧ (s ∨ t]) = ((a ∨ s) ∨ (a ∧ t]) = ((a] ∨ (s]) ∨ ((a] ∨ [t]).
Therefore (s] is a classic ideal.
Conversely, (s] is a classic ideal. Let x ≤ s ∨ t then (x] ⊆ (s ∨ t] = (s] ∨ [t].
So (x] = (x] ∨ ((s] ∧ [t]) = ((x] ∨ (s]) ∨ ((x] ∨ [t]).
Thus x ∈ ((x] ∨ (s]) ∨ ((x] ∨ [t]) which implies x ≤ a ∨ b ≤ x, where a ∈ (x] ∨ (s], b ∈ (x] ∨ [t]. Therefore x = a ∨ b, a ≤ s, b ≤ t. Hence s is a classic element.

Now, we give a characterization of classic ideal.

2.8 Theorem: An ideal T of S is a classic ideal if and only if x ≤ t ∨ a for some t ∈ T, x ∈ S implies there exists t1 ∈ T and a1 ≤ a such that x = t1 ∨ a1.

Proof : Suppose T is a classic ideal of S, then (x] ∨ (T ∧ [a]) = ((x] ∨ T) ∧ (x] ∨ [a]) for (x], [a] ∈ I(S). Let x ≤ t ∨ a for some t ∈ T, x ∈ S.
Then x ∈ (x] ∨ (T ∧ [a]) = ((x] ∨ T) ∧ (x] ∨ [a]) which implies x ≤ t1 ∨ a1 for some t1 ≤ x, t1 ∈ T and a1 ≤ a, implies t1 ∨ a1 ≤ x. Therefore x = t1 ∨ a1 with t1 ∈ T and
Suppose $T$ is a classic ideal and $x \in I \lor (T \land J)$ for any $I, J \in I(S)$, then $x \in I$ or $x \leq t \lor j$ for some $t \in T, j \in J$, then there exists $t_1 \in T$ and $j_1 \leq j$ such that $x = t_1 \lor j_1$. Thus $x \in (I \lor T) \land (I \lor J)$. Therefore $I \lor (T \land J) \subseteq (I \lor T) \land (I \lor J)$. Similarly we can prove that $(I \lor T) \land (I \lor J) \subseteq I \lor (T \land J)$. Hence $T$ is a classic ideal.

### 2.9 Theorem
For an ideal $T$ of $S$, the following conditions are equivalent.

- (i) $T$ is a classic ideal.
- (ii) For any ideal $I$, $T \land I = \{ t \lor i / t \in T, i \in I \}$. 
- (iii) For any principal ideal $I$, $T \land I = \{ t \lor i / t \in T, i \in I \}$. 
- (iv) For any $x, y \in S$, $(x \lor (T \land y)) = ((x \lor T) \land (x \lor y))$.

**Proof:**
- (i) $\Rightarrow$ (ii)
  Suppose $T$ is a classic ideal and $x \in T \land I$, then $x \in J, x \in I$ and $x \leq t \lor i$ for some $t \in T, i \in I$. Since $T$ is a classic ideal, then by above theorem 2.8, there exists $t_1 \in T$ such that $x = t_1 \lor i_1$.
  Thus for any $x \in T \land I$ we have $x = t_1 \lor i_1$ for $t_1 \in T$ and $i_1 \in I$.
  Therefore the condition (ii) holds.
- (ii) $\Rightarrow$ (iii)
  Suppose condition (ii) holds, for any ideal $I$, $T \land I = \{ t \lor i / t \in T, i \in I \}$.
  Let $I$ be any principal ideal, then for $a \in I, a \leq i$ for some $i \in I$, we have $a \lor i = i \in I$. Let $x \in T \land I$, then $x \in T$ and $x \in I$, where $I$ is a principal ideal.
  Then $x = t \lor i$ for $t \in T$ and $i \in I$. Therefore the condition holds obviously.
- (iii) $\Rightarrow$ (iv)
  Suppose condition (iii) holds; that is for any principal ideal $I$, we have $T \land I = \{ t \lor i / t \in T, i \in I \}$.
  To show that $(x \lor (T \land y)) = ((x \lor T) \land (x \lor y))$ for any $x, y \in S$.
  Let $p \in (x \lor (T \land y))$, then $p \leq x$ and $p = t \lor y_1$ for some $t \in T$ and $y_1 \in (y]$. Now $t \leq t \lor y_1$, then $t \leq p$ and $p \leq x$ implies $t \leq p \leq x$.
  Also $y_1 \leq t \lor y_1$, then $y_1 \leq p \leq x$. Thus $y_1 \leq x$, we have $y_1 \leq p$ and $p \leq x$.
  Then $y_1 \leq x$, thus we have $t \in (x \lor T)$ and $y_1 \in (x \lor y]$, implies $t \lor y_1 \in ((x \lor T) \land (x \lor y))$. Therefore $p \in ((x \lor T) \land (x \lor y))$.
  Hence condition (iv) holds.
- (iv) $\Rightarrow$ (i)
  Suppose (iv) holds, then $(x \lor (T \land y)) = ((x \lor T) \land (x \lor y))$ for any $x, y \in S$.
  To prove that $T$ is classic ideal.
Let $x \leq t \lor a$ for some $t \in T, x, a \in S$.
Then $x \in T \land (a]$, and so $(x] = (x) \lor (T \land (a]) = ((x] \lor T) \land ((x] \lor (a])$.
Then $x \leq p \lor q$ for some $p \in T, q \leq a$ with $p, q \leq x$. Thus $p \lor q \leq x$.
Therefore $x = p \lor q$ for $p \in T$ and $q \leq a$. Hence, by theorem 2.8 $T$ is a classic ideal.

### 2.10 Theorem
Let $I$ be an arbitrary ideal and $I$ a classic ideal of $S$. If $I \lor T$ and $I \land J$ are both principal then $I$ itself is principal.

**Proof:**
Let $I$ be an arbitrary ideal and $T$ is a classic ideal of $S$. Let $I \lor T$ and $I \land J$ are both principal.
Let $I \lor T = (a)$ and $I \land T = (b)$. Then $a = t \lor x$ for $t \in T$ and $x \in I$ and $b = t \land x$.
We claim that $I = (x \lor b]$.
Now $(a) = (t) \lor (x) \subseteq T \lor (x) \subseteq T \lor (x \lor b] \subseteq T \lor I = (a)$
So that $T \lor I = T \lor (x \lor b]$. Further $(b) = T \land I \supseteq T \land (x \lor b] \supseteq T \land (b) = (b)$.
So that $T \land I = T \land (x \lor b]$, Since $T$ is a classic ideal, we have $I = (x \lor b]$.
Therefore $I$ is a principal ideal.
The following theorem gives a characterization classic ideals in terms of a congruence.

2.11 Theorem: An ideal $T$ of semilattice $S$ is classic ideal if and only if
$\theta = \{(x, y) \in S \times S \mid x \lor t = y \lor t = x \lor y \text{ for some } t \in T\}$ is a congruence relation
and $(x, y) \in \theta$, $z \leq y$, which implies that there exists $w \leq x$, $z$ such that $(w, z) \in \theta$.

Proof: Let $T$ be a classic ideal.
First of all we consider that “If $\theta$ is a congruence relation then we define
$\theta_T = \{(x, y) \in S \times S \mid x \lor t = y \lor t \text{ for } t \in T\}”$ where $T$ is an ideal of $S$.
Clearly $\theta \subseteq \theta_T$. Let $(x, y) \in \theta_T$, so that $x \lor t = y \lor t$ for
$t \in T$. Since $T$ is a classic ideal, for $x \in ((x) \lor ((y) \land T)) = ((x) \lor (y)) \land ((x) \lor T)$.
Thus, there exists $y_1 \leq y$, $t_1$ with $t_1 \leq x$ where $t_1 \in T$, such that $x \leq y_1 \land t_1$ and
hence $x = y_1 \land t_1$. By a similar argument, there exists $x_1 \leq x$, $t_2$ where $t_2 \in T$ such that
$y = x_1 \lor t_2$. Therefore $x \lor y = x \lor t_1 \lor t_2 = y \lor t_1 \lor t_2$ for $t_1 \lor t_2 \in T$.
Therefore $(x, y) \in \theta$. Therefore $\theta_T \subseteq \theta$. Let $(x, y) \in \theta$ and $z \leq y$, then $x \lor t = y \lor t$ for
some $t \in T$; and hence $z \leq y \lor t \leq y \lor t = x \lor t$. Thus $z \in \{(z) \lor ((x) \land T)
= ((z) \lor (x)) \land ((z) \lor T)$. So that $z \leq x_1 \lor t_1$, where $x_1 \leq z$, $x; t_1 \leq z$ where $t_1 \in T$ and
hence $z = x_1 \lor t_1$. Also $(Z, x_1) \in \theta_T$.
Conversely, let $x \in A \lor (T \land B)$ so that $x \in A$ and $x \leq t \lor b$ for some $t \in T$
and $b \in B$. Since $t \lor b \lor t = b \lor t$, we have $(t \lor b, b) \in \theta$, then there exists $w \geq x$, $b$
such that $(w, x) \in \theta$. Hence $w \lor t_1 = x \lor t_1 = w \lor x = x$.
Therefore $w \lor t_1 = x$ for $t_1 \in T$. Therefore $T$ is a classic ideal.
Finally we include the following characterization of classic element .

2.12 Theorem: An element $s$ of semilattice is a classic element if and only if
(i) $s$ is distributive and (ii) $s$ is modular.
Proof: (i) Suppose $s \in S$ is a classic element, as “Every classic element of $S$ is a
distributive element of $S$”. Hence $s \in S$ is a distributive element.
(ii) Let $y \leq s \lor x$ with $x \leq y$. Since $s$ is a classic element we have $y = s_1 \lor x_1$ for some
$s_1 \leq s$, $x_1 \leq x$ for $s_1, x_1 \in S$. Since $x \leq y$ we have $y = x \lor y = x \lor s_1 \lor x_1 = s_1 \lor x$.
Therefore $y = s_1 \lor x$. Hence $s$ is modular. Conversely, let $y \leq s \lor x$ for $x, y \in S$. Since
$y \leq s \lor y$ and $s$ is distributive there exists $t \in T$ with $t \leq x, t \leq y$ such that $y \leq s \lor t$.
Then by modularity of $s$, there exists $s_1 \leq s$ such that $y = s_1 \lor t$, so for $y \leq s \lor x$,
$y = s_1 \lor t$ for some $s_1 \leq s$ and $t \leq x$. Hence $s$ is a classic element in $S$.

2.13 Theorem: In a modular semilattice, every distributive element is a classic element.
Proof: Suppose $S$ is a modular semilattice. Then for $x, y, z \in S$ with $z \leq x \leq y \lor z$
there exists $y_1 \in S$ with $y_1 \leq y$ such $x = y_1 \lor z$. Let $s \in S$ be a distributive element.
Let $x \leq s \lor y$ for some $x, y \in S$. Then there exists $z \in S$ such that $z \leq x, z \leq y$ and
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\[ x \leq s \lor z \text{.} \] As \( S \) is a modular semilattice every element of \( S \) is modular. Thus \( s \in S \) is modular then there exists \( s_1 \leq s \) such that \( x = s_1 \lor y_1 \). Hence \( s \) is a classic element.

References


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