Determination of Groups which Admit
a Particular Type of L-Fuzzy Subgroups

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Abstract. Groups which posses a L-fuzzy subgroup $\mu$, where L is a meet semi lattice satisfying the condition $\mu(x) \leq \mu(y)$ implies $\langle x \rangle \supseteq \langle y \rangle$ are determined.

Mathematics Subject Classification: 08A72, 20N25, 03E72

Keywords: Meet semi lattice, L-fuzzy subgroups, Cyclic groups of prime power order, Cyclic groups of square free order, Complete atomic lattice
1 Introduction

Mohamad Asaad in the paper [2] “Groups and fuzzy subgroups” characterized cyclic groups in the class of groups of prime power order which posses a fuzzy subgroup $\mu$ such that $\mu(x) \leq \mu(y) \Rightarrow \langle x \rangle \supseteq \langle y \rangle$. In fact he established the following theorem. If $G$ is a group of prime power order then $G$ is cyclic iff there exists a fuzzy subgroup $A$ of $G$ such that for any $x, y \in G$

\[(i) \ A(x) = A(y) \Rightarrow \langle x \rangle = \langle y \rangle,\]

\[(ii) \ A(x) > A(y) \Rightarrow \langle x \rangle \subset \langle y \rangle.\]

The objective of the authors is to give an elegant proof of the above theorem and to determine all groups which posses such a fuzzy subgroup. In the earlier papers [3],[4] the authors have provided partial answers to the above objective, and this paper provides a complete solution. Asaad’s study deals with only fuzzy subgroups taking values in [0,1]. Though [0,1] is a complete lattice, the absence of atoms makes it impossible to characterize all the groups which posses a fuzzy subgroup described above.

In order to remove this difficulty the authors consider the groups which posses L-fuzzy subgroups where $L$ is a meet semi lattice. Infact it is established that if $G$ is a group and $L$ is a meet semi lattice and $\mu : G \rightarrow L$ is a L-fuzzy subgroup of $G$ which satisfies the condition (1): $\mu(x) \leq \mu(y)$ implies $\langle x \rangle \supseteq \langle y \rangle$, then $G$ is cyclic iff $Im\mu$ has a least element.

2. Preliminaries

In this section, we collect some important definitions and results which are already proved in earlier papers[3] and [4], for ready reference.

We begin with the following.

**Definition 2.1:** Let $X$ be any non-empty set and let $(L, \wedge)$ be a semi lattice, then any mapping $\mu$ from $X$ into $L$ is called an L-fuzzy subset of $X$.

**Definition 2.2:** Let $G$ be a group. A L-fuzzy subset $\mu$ of $G$ is called a L-fuzzy subgroup of $G$ if (i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$, (ii) $\mu(x^{-1}) \geq \mu(x)$ for all $x$ in $G$.

**Theorem 2.3:** Let $G$ be a group, $L$ a meet semi lattice and $\mu : G \rightarrow L$ is a L-fuzzy subgroup of $G$, satisfying the condition(1). Then $G$ is cyclic iff $Im\mu$ has a least element.
Proof. Suppose $G$ is a cyclic group generated by $x$. For any $y$ in $G$, 
$y = x^m$ for some $m \in \mathbb{Z}$ so that $\mu(y) = \mu(x^m) \geq \mu(x)$, which implies that 
$\mu(x)$ is the least element of $\text{Im}\mu$. Conversely, if $\text{Im}\mu$ has the least element, 
$\mu(x)$ where $x$ is in $G$ then, this implies $\langle x \rangle \supseteq \langle y \rangle$ for all $y$ in $G$ so that $y \in \langle x \rangle$.

Hence follows that $G = \langle x \rangle$. 

\textbf{Theorem 2.4}:[4] If $\mu : G \to L$ is an $L$-fuzzy subgroup of $G$ satisfying condi-
tion(1) and $\text{Im}\mu$ has least element then $\text{Im}\mu$ is a complete atomic lattice.

\textbf{Theorem 2.5}:[4] Let $G$ be a group and $G$ admits an $L$-fuzzy subgroup $\mu$ satisfying condition(1). Then the following conditions are hold.

(a) $G \cong \mathbb{Z}$ iff $\text{Im}\mu$ contains an infinite number of atoms.
(b) $G \cong \mathbb{Z}_n$ iff $\text{Im}\mu$ contains a finite number of atoms.

3 Some characterizations of cyclic groups in terms of $L$-fuzzy sub-
groups.

In this section, $L$ stands for a meet semi lattice $(L, \wedge)$ with atleasat two 
elements.

We begin with the following.

\textbf{Theorem 3.1}: A group $G$ is a cyclic group of prime power order iff for 
some meet semi lattice $L$ and $L$-fuzzy subgroup $\mu : G \to L$ satisfying the 
condition(1), $\text{Im}\mu$ is a finite chain.

\textit{Proof.} Suppose $G$ is a cyclic group of prime power order, say $p^n$. We know 
that $[0, 1]$ is a meet semi lattice. Let $L = [0, 1]$. Now define a $L$-fuzzy subset 
$\mu : G \to L$ by $\mu(x) = a_i$ if $o(x) = p^i$, $i = 1, 2, ..., n$

and $\mu(e) = a_0$ where $a_0 > a_1 > ... > a_n$. Thus $\text{Im}\mu$ is a finite chain.

We now prove that $\mu$ is $L$-fuzzy subgroup of $G$. Let $x, y \in G$. Since $G$ is 
a cyclic group of prime power order, we have $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. So 
$\langle xy \rangle \subseteq \langle x \rangle$ or $\langle x \rangle \subseteq \langle x \rangle$. $\mu(xy) = a_i = a_i \wedge a_i \geq \mu(x) \wedge \mu(y)$, and clearly 
$\mu(x^{-1}) \geq \mu(x)$ for all $x$ in $G$. Therefore $\mu$ is a $L$-fuzzy subgroup of $G$. To 
show $\mu$ satisfies the condition (1), suppose that $\mu(x) = \mu(y)$. Then we have 
$\mu(x) = a_i = \mu(y)$ for some $i$ and $o(x) = p^i = o(y)$. Since $G$ is a cyclic
Proof. Suppose condition(1), with $\text{Im}\mu$ group of prime power order. Let $y$ this implies that $\langle \mu, \mu \rangle _p$ with $\mu \approx \mu$. But isomorphic to $\mu$. L exists a meet semi lattice $A$ group

Theorem 3.3: A group $G$ is a cyclic group of square free order if there exists a meet semi lattice $L$ and $\mu : G \rightarrow L$ is a L-fuzzy subgroup of $G$ satisfying the condition(1), with $\text{Im}\mu$ is a finite lattice.

Proof. Suppose $G$ is a finite cyclic group. Then $G \cong \mathbb{Z}_m$. We know that $P(S)$, with $S = \{1, 2, \ldots, n\}$ is a meet semi lattice. Let $L = P(S)$. Now define $\mu : G \rightarrow L$ by $\mu(x) = \{x\}$ The set of all positive divisors of $x$.

Clearly $\mu$ is a L-fuzzy subgroup of $G$ and satisfies the condition(1). For let $\mu(x) \leq \mu(y)$ implies $x \mid y$ then $\langle y \rangle \subseteq \langle x \rangle$. Since $G$ is finite cyclic group follows that $\text{Im}\mu$ is finite lattice.

Conversely, suppose $G$ admits a L-fuzzy subgroup of $G$ satisfying condition(1) such that $\text{Im}\mu$ is finite lattice. This implies $\text{Im}\mu$ has a least element and hence $G$ is a cyclic group. Also if $G \cong \mathbb{Z}$, then by theorem 2.5(a)[4], $\text{Im}\mu$ is infinite which is not the case. Thus $G$ is a finite cyclic group.

Theorem 3.3: A group $G$ is a cyclic group of square free order if there exists a meet semi lattice $L$ and a L-fuzzy subgroup $\mu : G \rightarrow L$ satisfying the condition(1) such that $\text{Im}\mu$ is a finite Boolean lattice.

Proof. Suppose $G$ is a cyclic group of square free order. We know that the subgroup lattice $S(G)$ is a Boolean lattice. Let $L = S(G)$. Now define $\mu :
$G \rightarrow L$ by $\mu(x) = \langle x \rangle'$, where $\langle x \rangle'$ is the Boolean complement of $\langle x \rangle$ in $L$. Clearly $\mu$ is a L-fuzzy subgroup as for any $x, y$ in $G$, $\mu(xy) = \langle xy \rangle' \geq \langle x \rangle' \land \langle y \rangle' = \mu(x) \land \mu(y)$ and $\mu(x^{-1}) = \langle x^{-1} \rangle' = \langle x \rangle' = \mu(x)$. Also $\mu$ satisfying the condition(1) i.e., $\mu(x) \leq \mu(y) \Rightarrow \langle x \rangle \supset \langle y \rangle$. Clearly $\mu$ is one-one and onto. Infact $\text{Im}\mu$ is a dual lattice of $L$ and hence it is a finite Boolean lattice.

Conversely, suppose $G$ admits a L-fuzzy subgroup $G$ satisfying condition(1) such that $\text{Im}\mu$ is a finite Boolean lattice. By theorem 2.3[4] follows that $G$ is cyclic and that the subgroup lattice $S(G)$ is a Boolean lattice. Because $S(G)$ is a Boolean lattice, follows that $G$ is finite, since the subgroup lattice of $\mathbb{Z}$ is not a Boolean lattice and also follows that $G$ is square free order.

**Theorem 3.4:** Let $G$ be a group. Then $G \cong \mathbb{Z}$ iff for some meet semi lattice $L$ and a L-fuzzy subgroup $\mu : G \rightarrow L$ satisfying the condition(1), $\text{Im}\mu$ is a complete atomic lattice with infinite number of atoms.

**Proof.** Let $G$ be a group. Suppose $G \cong \mathbb{Z}$. We know that $\mathbf{P}(\mathbb{N})$ is a meet semi lattice. Let $L = \mathbf{P}(\mathbb{N})$. Now define $\mu : G \rightarrow L$ by $\mu(x) =$ The set of all the positive divisors of $x$. Clearly $\mu$ is a L-fuzzy subgroup of $\mathbb{Z}$ and satisfies the condition (1). For let $\mu(x) \leq \mu(y)$ implies $x|y$ the $\langle y \rangle \subseteq \langle x \rangle$. Since $G$ is cyclic group, from theorem 2.3[4], $\text{Im}\mu$ has a least element. By theorem 2.4[4] follows that $\text{Im}\mu$ is a complete atomic lattice. Note that for any positive integer $x$ in $\mathbb{Z}$, $\mu(x)$ is an atom iff $x$ is prime. Thus $\text{Im}\mu$ contains infinite number of atoms.

Conversely, let $L$ be a meet semi lattice and $G$ admit a L-fuzzy subgroup of $G$ satisfying condition(1), with $\text{Im}\mu$ is a complete lattice, having infinite number of atoms. By theorem 2.5(a)[4] follows that $G$ is isomorphic to $\mathbb{Z}$.

Finally we conclude with the following theorem.

**Theorem 3.5:** A group $G$ is cyclic iff there exists a meet semi lattice $L$ and a L-fuzzy subgroup $\mu : G \rightarrow L$ satisfying the condition(1), with $\text{Im}\mu$ is a complete atomic lattice.

**Acknowledgements**: The authors are grateful to Prof. K.L.N. Swamy for his valuable suggestions and discussions on this work.
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Received: April, 2010