Strong $S_1$-Near Rings

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Abstract

In [3] we defined a right near ring $(N,+,\cdot)$ to be an $S_1$-near ring if for every $a \in N$, there exists $x \in N^*$, where $N^* = N - \{0\}$, such that $axa = xa$. In this paper we call $N$ a Strong $S_1$-near ring if for every $a \in N$, $\{x \in N^* | axa = xa \} = N^*$. We study some of its important properties, obtain a characterization and also a structure theorem under certain conditions.

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1 Introduction

Throughout this paper $N$ stands for a right near ring $(N,+,\cdot)$ and '0' denotes the identity element of $(N,+)$. A non-empty subset $A$ of $N$ is called a multiplicative system if $A$ is closed under multiplication [2]. $N$ is said to be regular if for every $a \in N$ there exists $b \in N$ such that $a = aba$. A map $f$ from $N$ into $N$ is called a mate function for $N$ if $a = af(a)a$ for all $a$ in $N$. $f(a)$ is called a mate of $a$ [4]. For basic concepts and terms used but not defined in this paper we refer to Pilz[1].
2 Notations

We freely make use of the following notations in this paper.

(a) \( E \) denotes the set of all idempotents of \( N \) (\( a \in E \) if and only if \( a^2 = a \)).
(b) \( N^* \) denotes the set of all non-zero elements of \( N \), i.e \( N^* = N - \{0\} \).
(c) \( N_0 = \{ a \in N \mid a0 = 0 \} \) - the zero symmetric part of \( N \) (\( N \) is called zero symmetric if \( N = N_0 \)).

3 Preliminary Results

We freely make use of the following results from [1] and [4] and designate them as \( R(1), R(2) \), etc.,

\( R(1) \) \( N \) is subdirectly irreducible if and only if the intersection of any family of non-zero ideals is again non-zero (Theorem 1.60, p.25 of [1]).
\( R(2) \) \( N \) has IFP (i.e Insertion of Factors Property) if for \( x,y \in N \), \( xy = 0 \Rightarrow xny = 0 \) for all \( n \in N \) (Definition 9.1, p.288 of [1]).
\( R(3) \) \( N \) has Strong IFP if for all ideals \( I \) of \( N \), \( ab \in I \Rightarrow anb \in I \) for all \( a,b,n \in N \) (Proposition 9.2, p.289 of [1]).
\( R(4) \) A zero symmetric near ring \( N \) has IFP if and only if \( (0:n) \) is an ideal for all \( n \in N \) (Theorem 9.3, p.289 of [1]).
\( R(5) \) \( N \) has Property(\( P_4 \)) if for all ideals \( I \) of \( N \), \( xy \in I \Rightarrow yx \in I \) (Definition 9.4, p.289 of [1]).
\( R(6) \) If \( N \) has IFP and if \( xy = 0 \Rightarrow yx = 0 \) for \( x,y \in N \) then we say that \( N \) has \((*,\text{IFP})\) (Lemma 2.3 of [4]).

4 \( S_1 \)-near rings and the subsets \( N_{S_1}(a) \), \( a \in N \)

As in [3] we have the following Definition.

**Definition 4.1** \( N \) is called an \( S_1 \)-near ring if for every \( a \in N \), there exists \( x \in N^* \) such that \( axa = xa \).

Throughout this paper we use the following notation.

**Notation 4.2** For any \( a \in N \), we denote \( \{ x \in N^* \mid axa = xa \} \) by \( N_{S_1}(a) \).

**Remark 4.3** It easily follows that \( N \) is an \( S_1 \)-near ring if and only if \( N_{S_1}(a) \neq \phi \) for all \( a \in N \).

The following examples substantiate this remark.
Examples 4.4 (a) Let \((N, +, \cdot)\) be the near ring where \((N, +)\) is the Klein’s four group \(N = \{0, a, b, c\}\) and the semigroup operation \(\cdot\) is defined as follows (Scheme(1), p.408 of Pilz[1]).

\[
\begin{array}{c|cccc}
\cdot & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & a & a \\
b & 0 & b & b & b \\
c & 0 & c & c & c \\
\end{array}
\]

Clearly this is an \(S_1\)-near ring. We observe that \(N_{S_1}(x) \neq \emptyset\) for all \(x \in N\).

\(N_{S_1}(0) = \{a, b, c\}\), \(N_{S_1}(a) = \{a\}\), \(N_{S_1}(b) = \{b\}\), \(N_{S_1}(c) = \{c\}\).

(b) We consider the near ring \((N, +, \cdot)\) where \((N, +)\) is the group of integers modulo 5 and \(\cdot\) is defined as follows (Scheme(6), p.408 of Pilz[1]).

\[
\begin{array}{c|cccc}
\cdot & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 1 & 0 & 0 \\
2 & 0 & 3 & 2 & 0 & 0 \\
3 & 0 & 2 & 3 & 0 & 0 \\
4 & 0 & 0 & 1 & 4 & 0 \\
\end{array}
\]

It is easy to see that this is not an \(S_1\)-near ring. It is worth nothing that \(N_{S_1}(2) = \emptyset\).

We furnish below a condition for an \(S_1\)-near ring to be regular.

**Proposition 4.5** Let \(N\) be an \(S_1\)-near ring. If \(a \in N_{S_1}(a) a\) for all \(a \in N\) then \(N\) is regular.

**Proof** Let \(a \in N\). By hypothesis \(a = xa\) for some \(x \in N_{S_1}(a)\). Since \(x \in N_{S_1}(a)\), \(axa = xa\). Therefore \(a = axa\). Thus \(N\) is regular.

**Remark 4.6** Converse of Proposition 4.5 is not valid. Consider the near ring \((N, +, \cdot)\) where \((N, +)\) is the group of integers modulo 6 and \(\cdot\) is defined as follows (Scheme (27), p.409 of Pilz[1]).

\[
\begin{array}{c|cccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 & 5 \\
2 & 0 & 2 & 4 & 0 & 2 & 4 \\
3 & 0 & 3 & 0 & 3 & 0 & 3 \\
4 & 0 & 4 & 2 & 0 & 4 & 2 \\
5 & 0 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
\]
This $S_1$-near ring is regular. But $2 \notin N_{S_1}(2)$ and $5 \notin N_{S_1}(5)$.

**Lemma 4.7** Let $N$ be an $S_1$-near ring. Then $N_{S_1}(a)$ has no non-zero zero divisors if and only if $N_{S_1}(a)$ is a multiplicative system.

**Proof** Since $N$ is an $S_1$-near ring, $N_{S_1}(a) \neq \emptyset$ for all $a \in N$.

For the 'only if' part, let $x,y \in N_{S_1}(a)$. Then $x,y \in N^*$ and $ax = xa$, $aya = ya$. It follows that $a(xy)a = ax(ya) = ax(aya) = (ax)ya = (xa)ya = x(aya) = x(ya) = (xy)a$. Further $N_{S_1}(a)$ has no non-zero zero divisors, $xy \neq 0$. Consequently $xy \in N_{S_1}(a)$. Thus $N_{S_1}(a)$ is a multiplicative system.

For the 'if' part, let $x,y \in N_{S_1}(a)$. Since $N_{S_1}(a)$ is a multiplicative system, $xy \in N_{S_1}(a)$. As $N_{S_1}(a) \subset N^*$, it follows that $xy \neq 0$ and hence $N_{S_1}(a)$ has no non-zero zero divisors.

The following is a simple characterization of zero symmetric near ring.

**Proposition 4.8** $N$ is zero symmetric if and only if $N^* = N_{S_1}(0)$.

**Proof** For the 'only if' part, let $x \in N^*$. Since $N = N_0$, $x0 = 0 \Rightarrow 0x0 = 0 = x0 \Rightarrow x \in N_{S_1}(0)$. Therefore $N^* \subset N_{S_1}(0)$. Clearly then $N^* = N_{S_1}(0)$.

For the 'if' part we observe that $N^* = N_{S_1}(0) \Rightarrow 0x0 = x0$ for all $x \in N^* \Rightarrow x0 = 0$ for all $x \in N^*$. Consequently, $N$ is zero symmetric.

### 5 Strong $S_1$-Near Rings

In this section we define the concept of Strong $S_1$-near rings, study some of its important properties, obtain a structure theorem and also a characterization of such near rings.

**Definition 5.1** A near ring $N$ is said to be a Strong $S_1$-near ring if $N^* = N_{S_1}(a)$ for all $a \in N$.

**Examples 5.2** (a) Let $(N,+)$ be the Klein’s four group with $N = \{0,a,b,c\}$ and we define ‘.’ as follows (Scheme(7), p-408 of Pilz[1]).

<table>
<thead>
<tr>
<th>.</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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<tr>
<td>0</td>
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<td>a</td>
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<td>c</td>
<td>0</td>
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The near ring $(N,+.,.)$ is a Strong $S_1$-near ring (In general, a commutative and Boolean near ring is a Strong $S_1$-near ring).

(b) Let $(N,+)$ be the Symmetric group of degree 3 with $N = \{0,a,b,c,x,y\}$ and we define ‘.’ as follows (Scheme(37), p.411 of Pilz[1])
This near ring is a Strong $S_1$-near ring. It is worth noting that it is not regular.

**Proposition 5.3** $N$ is a Strong $S_1$-near ring if and only if $axa = xa$ for all $a \in N$ and for all $x \in N^*$.

**Proof** is straightforward.

The following Corollary is an immediate consequence of Proposition 5.3 and Definition 4.1

**Corollary 5.4** Every Strong $S_1$-near ring is an $S_1$-near ring

**Remark 5.5** Converse of Corollary 5.4 is not valid. For example, consider the near ring $(N, +, \cdot)$ where $(N, +)$ is the Klein’s four group with $N = \{0, a, b, c\}$ and $\cdot$ is defined as follows (Scheme(9), p.408 of Pilz[1])

<table>
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<tr>
<th>$\cdot$</th>
<th>0</th>
<th>a</th>
<th>b</th>
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<th>x</th>
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<td>c</td>
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<td>b</td>
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<td>a</td>
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<td>c</td>
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We observe that this is an $S_1$-near ring; however it is not a Strong $S_1$-near ring (since $cbc \neq bc$).

**Proposition 5.6** If $N$ is a Strong $S_1$-near ring then $N$ is zero symmetric.

**Proof** Since $N$ is a Strong $S_1$-near ring, from Proposition 5.3, $axa = xa$ for all $a \in N$ and for all $x \in N^*$. Putting $a = 0$ we get $0x0 = x0$ for all $x \in N^*$ and this implies $x0 = 0$ for all $x \in N^*$ and the desired result now follows.

**Remark 5.7** The example given under Remark 5.5 shows that the converse of Proposition 5.6 is not valid.

We furnish below a characterization of Strong $S_1$-near rings.
Theorem 5.8 N is a Strong $S_1$-near ring if and only if $axa = xa$ for all $a, x \in N$.

Proof If $N$ is a Strong $S_1$-near ring then from Proposition 5.6 $N$ is zero symmetric $\Rightarrow a0 = 0$ for all $a \in N \Rightarrow a0a = 0 = 0a$ for all $a \in N$. The rest of the proof is taken care of by Proposition 5.3.

We prove some properties of Strong $S_1$-near ring in the following Theorem.

Theorem 5.9 Let $N$ be a Strong $S_1$-near ring. Then
(i) $ab$ and $ba \in E$ for all $a, b \in N$.
(ii) $N$ has (*,IFP).
(iii) $N$ has Strong IFP.
(iv) $N$ has Property (P4).

Proof Let $N$ be a Strong $S_1$-near ring. Then it follows from Theorem 5.8 that $axa = xa$ for all $a, x \in N$ ..............(1)
(i) Let $a, b \in N$. Now (1) implies that $ab = bab = (ba)b = (aba)b = (ab)^2 \Rightarrow ab \in E$. In a similar fashion we get $ba \in E$.
(ii) Suppose $xy = 0$ for $x, y \in N$..........(2).
Now $yx = x(xy) = (x(yx)) = (xy)x = 0x \Rightarrow 0$.
Also for every $n \in N$, $xny = x(ny) = x(ny)$ by (1) $= (xy)ny = 0ny = 0$. Thus $N$ has (*,IFP).
(iii) Let $I$ be an ideal of $N$ and let $ab \in I$. Proposition 5.6 guarantees that $N$ is zero symmetric. Therefore $NI \subset I$ ....(3) and $IN \subset I$ ...(4).
Now $anb = (an)b$ by (1) $= (na(nb) = na(nbn)$ by (1) $= n(ab)nb \in I$ by (3) and (4).
From $R(3)$ it follows that $N$ has Strong IFP.
(iv) Let $I$ be an ideal of $N$ and let $xy \in I$. As in (iii) above $IN \subset I$ and $NI \subset I$. Now $(yx)^2 = yxyx = y(xy)x \in NI = (NI)N \subset IN \subset I$. Also $(yx)^2 \in I$.
Appealing to (i) we get $yx = (yx)^2 \in I$. Consequently $N$ has (P4).

With a view to establishing a structure theorem for Strong $S_1$-near rings, we prove the following Theorems.

Theorem 5.10 Any homomorphic image of a Strong $S_1$-near ring is a Strong $S_1$-near ring.

Proof is straightforward.

Theorem 5.11 Every Strong $S_1$-near ring is isomorphic to a subdirect product of subdirectly irreducible Strong $S_1$-near rings.

Proof By Theorem 1.62, p.26 of Pilz[1], $N$ is isomorphic to a subdirect product of subdirectly irreducible near rings $N_i$’s say and each $N_i$ is a homomorphic image of $N$ under projection map $\pi_i$. The desired result now follows from Theorem 5.10.
**Theorem 5.12** Let $N$ be a Strong $S_1$-near ring with mate function $f$. Then $N$ is subdirectly irreducible if and only if $N$ is Simple.

**Proof** Suppose $N$ is subdirectly irreducible. First we prove that for any non-zero idempotent $e$ in $N$, $(0:e) = \{0\}$. Let $D = \{e \in E - \{0\} | (0 : e) \neq \{0\}\}$. Suppose $D \neq \phi$. Let $B = \bigcap_{e \in D} (0 : e)$. Now Theorem 5.9 demands that $N$ has ($\ast$,IFP). From Proposition 5.6 and $R(4)$ we see that $(0:e)$ is an ideal. Since $N$ is subdirectly irreducible, $R(1)$ shows that $B \neq \{0\}$.

Let $a \in B - \{0\} \Rightarrow ae = 0$ for all $e \in D$ ............ (1)

Now $f(a)ae = f(a)0 = 0$ [Since $N = N_0$] $\Rightarrow ef(a)a = 0 \Rightarrow e \in (0 : f(a)a) \Rightarrow f(a)a \in D \Rightarrow af(a)a = 0$ [by (1)] $\Rightarrow a = 0$ which is a contradiction to $a \neq 0$.

Consequently for any non-zero idempotent $e$ in $N$, $(0:e) = \{0\}$. Since $N$ is a Strong $S_1$-near ring, from Theorem 5.8 we get $exe = xe \Rightarrow (ex-x)e = 0 \Rightarrow ex-x \in (0:e) = \{0\} \Rightarrow ex = x$ for all $x \in N$. i.e $x = ex \in Nx \Rightarrow N = Nx$ for all $x \in N$. Thus $N$ is Simple.

Converse is obvious.

We conclude our discussion with the following structure theorem for Strong $S_1$-near rings.

**Theorem 5.13** Every Strong $S_1$-near ring with a mate function is isomorphic to a subdirect product of Simple near rings.

**Proof** Collecting the pieces proved in Theorems 5.11 and 5.12 we get the desired result.

**References**


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