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Strong S_1-Near Rings

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Abstract

In [3] we defined a right near ring (N,+,·) to be an S_1-near ring if for every a ∈ N, there exists x ∈ N^*, where N^* = N - {0}, such that axa = xa. In this paper we call N a Strong S_1-near ring if for every a ∈ N, {x ∈ N^*| axa = xa } = N^*. We study some of its important properties, obtain a characterization and also a structure theorem under certain conditions.

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1 Introduction

Throughout this paper N stands for a right near ring (N,+,·) and '0' denotes the identity element of (N,+). A non-empty subset A of N is called a multiplicative system if A is closed under multiplication [2]. N is said to be regular if for every a ∈ N there exists b ∈ N such that a = aba. A map f from N into N is called a mate function for N if a = af(a)a for all a in N. f(a) is called a mate of a [4]. For basic concepts and terms used but not defined in this paper we refer to Pilz[1].

2 Notations

We freely make use of the following notations in this paper.

(a) E denotes the set of all idempotents of N (a $\in$ E if and only if a^2 = a).
(b) N^* denotes the set of all non-zero elements of N, i.e N^* = N - $\{0\}$.
(c) N_0 = $\{$a $\in$ N $|$ a0 = 0$\}$ - the zero symmetric part of N (N is called zero symmetric if N = N_0).

3 Preliminary Results

We freely make use of the following results from [1] and [4] and designate them as **R(1)**,**R(2)**..etc.,

R(1) N is subdirectly irrreducible if and only if the intersection of any family of non-zero ideals is again non-zero (Theorem 1.60, p.25 of [1]).
R(2) N has IFP (i.e Insertion of Factors Property) if for x,y $\in$ N, xy = 0 $\Rightarrow$ xny = 0 for all n $\in$ N (Definition 9.1, p.288 of [1]).
R(3) N has Strong IFP if for all ideals I of N, ab $\in$ I $\Rightarrow$ anb $\in$ I for all a,b,n $\in$ N (Proposition 9.2, p.289 of [1]).
R(4) A zero symmetric near ring N has IFP if and only if (0:n) is an ideal for all n $\in$ N (Theorem 9.3, p.289 of [1]).
R(5) N has Property(P_4) if for all ideals I of N, xy $\in$ I $\Rightarrow$ yx $\in$ I (Definition 9.4, p.289 of [1])
R(6) If N has IFP and if xy = 0 $\Rightarrow$ yx = 0 for x,y $\in$ N then we say that N has (*,IFP) (Lemma 2.3 of [4]).

4 S_1-near rings and the subsets N_{S_1}(a), a $\in$ N

As in [3] we have the following Definition.

Definition 4.1 *N is called an S_1-near ring if for every $a \in N$, there exists $x \in N^*$ such that $axa = xa$.*

Throughout this paper we use the following notation.

Notation 4.2 *For any $a \in N$, we denote $\{x \in N^* | \ axa = xa\}$ by $N_{S_1}(a)$.*

Remark 4.3 *It easily follows that N is an S_1-near ring if and only if $N_{S_1}(a) \neq \phi$ for all $a \in N$.*

The following examples substantiate this remark.

Examples 4.4 **(a)** *Let (N,+,·) be the near ring where (N,+) is the Klein's four group $N = \{0,a,b,c\}$ and the semigroup operation '·' is defined as follows (Scheme(1), p.408 of Pilz[1]).*

·	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Clearly this is an S_1-near ring. We observe that $N_{S_1}(x) \neq \phi$ for all $x \in N$. ($N_{S_1}(0) = \{a,b,c\}$, $N_{S_1}(a) = \{a\}$, $N_{S_1}(b) = \{b\}$, $N_{S_1}(c) = \{c\}$).

(b) *We consider the near ring (N,+,·) where (N,+) is the group of integers modulo 5 and '·' is defined as follows (Scheme(6), p.408 of Pilz[1]).*

·	0	1	2	3	4
0	0	0	0	0	0
1	0	0	4	1	0
2	0	0	3	2	0
3	0	0	2	3	0
4	0	0	1	4	0

It is easy to see that this is not an S_1-near ring. It is worth nothing that $N_{S_1}(2) = \phi$.

We furnish below a condition for an S_1-near ring to be regular.

Proposition 4.5 *Let N be an S_1-near ring. If $a \in N_{S_1}(a)a$ for all $a \in N$ then N is regular.*

Proof *Let $a \in N$. By hypothesis $a = xa$ for some $x \in N_{S_1}(a)$. Since $x \in N_{S_1}(a)$, $axa = xa$. Therefore $a = axa$. Thus N is regular.*

Remark 4.6 *Converse of Proposition 4.5 is not valid. Consider the near ring (N,+,·) where (N,+) is the group of integers modulo 6 and '·' is defined as follows (Scheme (27), p.409 of Pilz[1])*

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

This S_1-near ring is regular. But $2 \notin N_{S_1}(2)2$ and $5 \notin N_{S_1}(5)5$.

Lemma 4.7 *Let N be an S_1-near ring. Then $N_{S_1}(a)$ has no non-zero zero divisors if and only if $N_{S_1}(a)$ is a multiplicative system.*

Proof *Since N is an S_1-near ring, $N_{S_1}(a) \neq \phi$ for all $a \in N$.*
For the 'only if' part, let $x, y \in N_{S_1}(a)$. Then $x, y \in N^*$ and $axa = xa$, $aya = ya$. It follows that $a(xy)a = ax(ya) = ax(aya) = (axa)ya = (xa)ya = x(aya) = x(ya) = (xy)a$. Further $N_{S_1}(a)$ has no non-zero zero divisors, $xy \neq 0$. Consequently $xy \in N_{S_1}(a)$. Thus $N_{S_1}(a)$ is a multiplicative system.
For the 'if' part, let $x, y \in N_{S_1}(a)$. Since $N_{S_1}(a)$ is a multiplicative system, $xy \in N_{S_1}(a)$. As $N_{S_1}(a) \subset N^*$, it folows that $xy \neq 0$ and hence $N_{S_1}(a)$ has no non-zero zero divisors.

The following is a simple characterization of zero symmetric near ring.

Proposition 4.8 *N is zero symmetric if and only if $N^* = N_{S_1}(0)$.*

Proof *For the 'only if' part, let $x \in N^*$. Since $N = N_0$, $x0 = 0 \Rightarrow 0x0 = 0 = x0 \Rightarrow x \in N_{S_1}(0)$. Therefore $N^* \subset N_{S_1}(0)$. Clearly then $N^* = N_{S_1}(0)$.*
For the 'if' part we observe that $N^* = N_{S_1}(0) \Rightarrow 0x0 = x0$ for all $x \in N^* \Rightarrow x0 = 0$ for all $x \in N^*$. Consequently, N is zero symmetric.

5 Strong S_1-Near Rings

In this section we define the concept of Strong S_1-near rings, study some of its important properties, obtain a sructure theorem and also a characterization of such near rings.

Definition 5.1 *A near ring N is said to be a Strong S_1-near ring if $N^* = N_{S_1}(a)$ for all $a \in N$.*

Examples 5.2 **(a)** *Let $(N,+)$ be the Klein's four group with $N = \{0, a, b, c\}$ and we define '$\cdot$' as follows (Scheme(7), p-408 of Pilz[1]).*

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

The near ring $(N,+,\cdot)$ is a Strong S_1-near ring (In general,a commutative and Boolean near ring is a Strong S_1-near ring).

(b) *Let $(N,+)$ be the Symmetric group of degree 3 with $N = \{0, a, b, c, x, y\}$ and we define '$\cdot$' as follows (Scheme(37), p.411 of Pilz[1])*

$\cdot$	0	a	b	c	x	y
0	0	0	0	0	0	0
a	0	a	b	c	0	0
b	0	a	b	c	0	0
c	0	a	b	c	0	0
x	0	0	0	0	0	0
y	0	0	0	0	0	0

This near ring is a Strong S_1-near ring. It is worth nothing that it is not regular.

Proposition 5.3 *N is a Strong S_1-near ring if and only if $axa = xa$ for all $a \in N$ and for all $x \in N^*$.*

Proof *is straightforward.*

The following Corollary is an immediate consequence of Proposition 5.3 and Definition 4.1

Corollary 5.4 *Every Strong S_1-near ring is an S_1-near ring*

Remark 5.5 *Converse of Corollary 5.4 is not valid. For example, consider the near ring $(N,+,\cdot)$ where $(N,+)$ is the Klein's four group with $N = \{0, a, b, c\}$ and $\cdot$.' is defined as follows (Scheme(9), p.408 of Pilz[1])*

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	b
c	0	a	0	c

We observe that this is an S_1-near ring; however it is not a Strong S_1-near ring (since $cbc \neq bc$).

Proposition 5.6 *If N is a Strong S_1-near ring then N is zero symmetric.*

Proof *Since N is a Strong S_1-near ring, from Proposition 5.3, $axa = xa$ for all $a \in N$ and for all $x \in N^*$. Putting $a = 0$ we get $0x0 = x0$ for all $x \in N^*$ and this implies $x0 = 0$ for all $x \in N^*$ and the desired result now follows.*

Remark 5.7 *The example given under Remark 5.5 shows that the converse of Proposition 5.6 is not valid.*

We furnish below a characterization of Strong S_1-near rings.

Theorem 5.8 *N is a Strong S_1-near ring if and only if $axa = xa$ for all a,x ∈ N.*

Proof *If N is a Strong S_1-near ring then from Proposition 5.6 N is zero symmetric ⇒ $a0 = 0$ for all $a \in N \Rightarrow a0a = 0 = 0a$ for all $a \in N$. The rest of the proof is taken care of by Proposition 5.3.*

We prove some properties of Strong S_1-near ring in the following Theorem.

Theorem 5.9 *Let N be a Strong S_1-near ring. Then*
(i) *ab and ba ∈ E for all a,b ∈ N.*
(ii) *N has (*,IFP).*
(iii) *N has Strong IFP.*
(iv) *N has Property (P_4).*

Proof *Let N be a Strong S_1-near ring .Then it follows from Theorem 5.8 that $axa = xa$ for all a,x ∈ N ....(1)*
(i) *Let a,b ∈ N. Now (1) implies that $ab = bab = (ba)b = (aba)b = (ab)^2 \Rightarrow ab \in E$. In a similar fashion we get $ba \in E$.*
(ii) *Suppose $xy = 0$ for x,y ∈ N..........(2).*
Now $yx = xyx$ [by(1)] $= (xy)x = 0x$ [by(2)] $= 0$. Also for every $n \in N$, $xny = x(ny) = x(yny)$[by(1)] $= (xy)ny = 0ny = 0$. Thus N has (,IFP).*
(iii) *Let I be an ideal of N and let $ab \in I$. Proposition 5.6 guarantees that N is zero symmetric. Therefore $NI \subset I$ ....(3) and $IN \subset I$....(4). Now $anb = (an)b = (nan)b$ [by(1)] $= na(nb) = na(bnb)$ [by(1)] $= n(ab)nb \in I$ [by (3) and (4)]. From R(3) it follows that N has Strong IFP.*
(iv) *Let I be an ideal of N and let $xy \in I$. As in (iii) above $IN \subset I$ and $NI \subset I$. Now $(yx)^2 = yxyx = y(xy)x \in NIN = (NI)N \subset IN \subset I$. i.e $(yx)^2 \in I$. Appealing to (i) we get $yx = (yx)^2 \in I$. Consequently N has (P_4).*

With a view to establishing a structure theorem for Strong S_1-near rings, we prove the following Theorems.

Theorem 5.10 *Any homomorphic image of a Strong S_1-near ring is a Strong S_1-near ring.*

Proof *is straightforward.*

Theorem 5.11 *Every Strong S_1-near ring is isomophic to a subdirect product of subdirectly irreducible Strong S_1-near rings.*

Proof *By Theorem 1.62, p.26 of Pilz[1], N is isomorphic to a subdirect product of subdirectly irreducible near rings N_i's say and each N_i is a homomorphic image of N under projection map π_i. The desired result now follows from Theorem 5.10.*

Theorem 5.12 *Let N be a Strong S_1-near ring with mate function f. Then N is subdirectly irreducible if and only if N is Simple.*

Proof *Suppose N is subdirectly irreducible. First we prove that for any non zero idempotent e in N, (0:e) = $\{0\}$. Let $D = \{e \in E - \{0\} | (0 : e) \neq \{0\}\}$. Suppose $D \neq \phi$.Let $B = \bigcap_{e \in D} (0 : e)$. Now Theorem 5.9 demands that N has (*,IFP). From Proposition 5.6 and R(4) we see that (0:e) is an ideal. Since N is subdirectly irrreducible, R(1) shows that $B \neq \{0\}$.*
Let $a \in B - \{0\} \Rightarrow$ ae = 0 for all $e \in D$ .... (1)
Now f(a)ae = f(a)0 = 0 [Since $N = N_0$] $\Rightarrow$ ef(a)a = 0 $\Rightarrow$ e $\in$ (0:f(a)a) $\Rightarrow$ f(a)a $\in D \Rightarrow$ af(a)a = 0 [by(1)] $\Rightarrow$ a = 0 which is a contradiction to a $\neq$ 0. Consequently for any non-zero idempotent e in N, (0:e) = $\{0\}$. Since N is a Strong S_1-near ring, from Theorem 5.8 we get exe = xe $\Rightarrow$ (ex-x)e = 0 $\Rightarrow$ ex-x $\in$ (0:e) = $\{0\}$ $\Rightarrow$ ex = x for all $x \in N$. i.e $x = ex \in Nx \Rightarrow N = Nx$ for all $x \in N$. Thus N is Simple.
Converse is obvious.

We conclude our discussion with the following structure theorem for Strong S_1-near rings.

Theorem 5.13 *Every Strong S_1-near ring with a mate function is isomorphic to a subdirect product of Simple near rings.*

Proof *Collecting the pieces proved in Theorems 5.11 and 5.12 we get the desired result.*

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