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Almost Excellent Extensions and the FP-Homological Property

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Abstract

Let R be a ring and S an almost excellent extension of R. In this paper, we discussed the homological properties of FP-projective modules and FP-injective modules under the extension of rings, which mainly including almost excellent extensions, Polynomial extensions, Morita equivalences and localizations.

Keywords: excellent extensions; FP-projective modules; FP-injective modules

1 Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary.

Let us recall the definition of FP-injective modules. Let R be a ring, a left R-module N is called FP-injective (Madox, 1967; Stenstorm, 1970), if $Ext_R^1(F, N) = 0$ for any finitely presented left R-module F. The concepts of FP-injective dimension of N is defined as $inf\{n| \ Ext_R^{n+1}(F, N) = 0$ for any finitely presented left R-module $F\}$, and denoted by $FP\text{-}id(N)$. If no such n exists, set $FP\text{-}id(N) = \infty$. In Mao and Ding(2005), the FP-projective dimension of a left R-module M is defined as $fpd(M) = inf\{n| \ Ext_R^{n+1}(M, N) = 0$ for any FP-injective left R-module $N\}$. If no such n exists, set $fpd(M) = \infty$. M is called FP-projective, if $fpd(M) = 0$. i.e., $Ext_R^1(M, N) = 0$ for any FP-injective left R-module N.

The extension of rings and modules is one of the basic problems in the research of the ring and categories of modules, and excellent extension is an important extension of rings, which including finite matrix rings $M_n(R)$ and crossed product $R * G$, where the finite group G satisfies the condition $|G|^{-1} \in R$.

Suppose that R is a subring of a ring S, and that R and S have the same identity.
(1) The ring S is called right R-projective (Passman, 1977), if N_S is a submodule of M_S and $N_R \mid M_R$, then $N_S \mid M_S$. Where the notation $N_R \mid M_R$ means that N_R is a direct summand of M_R.
(2) A ring S is a finite normalizing extension of a ring R (Resco, 1981), if there exists a finite set $\{s_1, s_2, ..., s_n\} \subseteq S$ such that $S = \sum_{i=1}^{n} s_i R$ and $s_i R = R s_i$ for all $i = 1, 2, ..., n$.
(3) A finite normalizing extension $S \geq R$ is an almost excellent extension (Xue, 1977), if R is right R-projective, ${}_R S$ is flat and S_R is projective.
(4) An almost excellent extension $S \geq R$ is called excellent extension (Passman,
1977), if both ${}_R S$ and S_R is free with basis $\{s_1, s_2, ..., s_n\}$.

The notion of almost excellent extension was introduced and studied in [6] as a non-trivial generalization of excellent extensions.

2 Main Results

In the following, we mainly consider the properties of FP-projective modules and FP-injective modules under an almost excellent extensions of rings.

Lemma 2.1 [6] *Let S be an almost excellent extension of R and M_S a right R-module. Then*

(1) M_S is isomorphism to a summand of $(M \otimes_R S)_S$.
(2) M_S is isomorphism to a summand of $Hom_R(S, M)_S$.

Lemma 2.2 [4] *Let $\{M_i\}$ be a family of left R-modules. Then direct product $\prod M_i$ (respectively direct sum $\coprod M_i$) is left FP-injective iff each M_i is left FP-injective.*

Lemma 2.3 [5] *If S is an almost excellent extension of R, then the following are equivalent for any right S-module N:*

(1) N is an FP-injective right S-module.
(2) N is an FP-injective right R-module.
(3) $Hom_R(S, N)$ is an FP-injective right S-module.

Theorem 2.4 *If S is an almost excellent extension of R, then the following are equivalent for any right S-module M:*

(1) M is an FP-projective right R-module.
(2) $M \otimes_R S$ is an FP-projective right R-module.

(3) $M \otimes_R S$ *is an* FP*-projective right* S*-module.*
(4) M *is an* FP*-projective right* S*-module.*

Proof (1) $\Rightarrow$ (3) *Let* N *is an* FP*-injective right* S*-module. Then* N *is an* FP*-injective right* R*-module by Lemma 2.3. Since* $Ext^1_S(M \otimes_R S, N) \cong Ext^1_R(M, Hom_S(S, N)) \cong Ext^1_R(M, N)$*, we have* $Ext^1_R(M, N) = 0$ *by (1), therefore* $Ext^1_S(M \otimes_R S, N) = 0$.

(3) $\Rightarrow$ (4) *Note that* M_S *is isomorphic to a direct summand of* $M \otimes_R S$*, so* M *is an* FP*-projective right* S*-module by Lemma 2.2.*

(4) $\Rightarrow$ (1) *Let* N *be an* FP*-injective right* R*-module. Then* $Hom_R(S, N)$ *is an* FP*-injective right* S*-module by Lemma 2.3. we have* $Ext^1_S(M, Hom_R(S, N)) = 0$ *by (4). Note that* $Ext^1_R(M, N) \cong Ext^1_R(M \otimes_S S, N) \cong Ext^1_S(M, Hom_R(S, N))$*, therefore* $Ext^1_R(M, N) = 0$.

(2) $\Leftrightarrow$ (3) *is clear by the proof of* (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).

As an immediate corollary to Lemma 2.3 and Theorem 2.4, we have the following results.

Corollary 2.5 *Let* $R * G$ *be crossed product, where* G *is a finite group with* $|G|^{-1} \in R$*. Then the following are equivalent for any right* $(R * G)$*-module* N*:*

(1) N *is an* FP*-injective right* R*-module.*
(2) $Hom_R(R * G, N)$ *is an* FP*-injective right* R*-module.*
(3) $Hom_R(R * G, N)$ *is an* FP*-injective right* $(R * G)$*-module.*
(4) N *is an* FP*-injective right* $(R * G)$*-module.*

Corollary 2.6 *Let* $R * G$ *be crossed product, where* G *is a finite group with* $|G|^{-1} \in R$*. Then the following are equivalent for any right* $(R * G)$*-module* M*:*

(1) M *is an* FP*-projective right* R*-module.*
(2) $M \otimes_R (R * G)$ *is an* FP*-projective right* R*-module.*
(3) $M \otimes_R (R * G)$ *is an* FP*-projective right* $(R * G)$*-module.*
(4) M *is an* FP*-projective right* $(R * G)$*-module.*

Corollary 2.7 *Let* R *be a ring and* n *any positive integer. Then the following are equivalent for any right* $M_n(R)$*-module* N*:*

(1) N *is an* FP*-injective right* R*-module.*
(2) $Hom_R(M_n(R), N)$ *is an* FP*-injective right* R*-module.*
(3) $Hom_R(M_n(R), N)$ *is an* FP*-injective right* $M_n(R)$*-module.*
(4) N *is an* FP*-injective right* $M_n(R)$*-module.*

Corollary 2.8 *Let* R *be a ring and* n *any positive integer. Then the following are equivalent for any right* $M_n(R)$*-module* M*:*

(1) M *is an* FP*-projective right* R*-module.*
(2) $M \otimes_R M_n(R)$ *is an* FP*-projective right* R*-module.*

(3) $M \otimes_R M_n(R)$ is an FP-projective right $M_n(R)$-module.
(4) M is an FP-projective right $M_n(R)$-module.

The next Theorem is an interesting result about FP-injective dimension.

Theorem 2.9 *Let R and S be right coherent rings and S an almost excellent extension of R. Then*

$$FP\text{-}id_R(M) = FP\text{-}id_S(M) = FP\text{-}id_S(Hom_R(S, M)).$$

Proof *We have $FP\text{-}id_R(M) = FP\text{-}id_S(M)$ by [2, Theorem 3.2].*

It suffices to prove that $FP\text{-}id_S(M) = FP\text{-}id_S(Hom_R(S, M))$. First we claim that $FP\text{-}id_S(Hom_R(S, M)) \leq FP\text{-}id_S(M)$. For that, we may assume that $FP\text{-}id_S(M) = n < \infty$, then $FP\text{-}id_R(M) = n < \infty$, thus there exists an exact sequence of right R-modules $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to E_n \to 0$, where each E_i is an FP-injective. Since S_R is projective, we have the following exact sequence $0 \to Hom_R(S, M) \to Hom_R(S, E_0) \to Hom_R(S, E_1) \to \cdots \to Hom_R(S, E_{n-1}) \to Hom_R(S, E_n) \to 0$. Note that each $Hom_R(S, E_i)$ is FP-injective right S-module by Lemma 2.3, and so $FP\text{-}id_S(Hom_R(S, M)) \leq n$.

On the other hand, since M_S is isomorphism to a summand of $Hom_R(S, M)$, we have $FP\text{-}id_S(M) \leq FP\text{-}id_S(Hom_R(S, M))$, and this gives the desired result.

Next, we are concerned with the properties of FP-projective modules and FP-injective modules under the equivalence of category.

Proposition 2.10 *Let R and S be equivalent rings, $F : Mod\text{-}R \to Mod\text{-}S$ and $G : Mod\text{-}S \to Mod\text{-}R$ are equivalent functors. Then*

(1) N is an FP-injective right R-module if and only if $F(N)$ is an FP-injective right S-module.
(2) M is an FP-projective right R-module if and only if $F(M)$ is an FP-projective right S-module.

Proof *(1) ($\Rightarrow$) Let A be a finitely presented right S-module. Since the notions of finitely generated, projectivity and exact sequences are Morita invariants. Then $G(A)$ is a finitely presented right R-module. we have $Ext^1_R(G(A), N) = 0$ for any FP-injective right R-module N, and so it follows that $Ext^1_S(A, F(N)) = 0$ by the isomorphism $Ext^1_S(A, F(N)) \cong Ext^1_R(G(A), N)$, thus $F(N)$ is an FP-injective right S-module.*

($\Leftarrow$) is clear by $G(F(N)) \cong N$.

(2) ($\Rightarrow$) Let N be any FP-injective right S-module. Then $G(N)$ is an FP-injective right R-module by (1). we have $Ext^1_R(M, G(N)) = 0$ since M is an FP-projective right R-module. Note that $Ext^1_S(F(M), N) \cong Ext^1_R(M, G(N))$,

therefore $Ext^1_S(F(M),N)=0$, *and hence* $F(M)$ *is an* FP*-projective right* S*-module.*

($\Longleftarrow$) *is clear by* $G(F(M))\cong M$.

According to this Proposition, it is easy to get the following corollary.

Corollary 2.11 *Let* R *be ring,* $e\in R$ *is an nonzero idempotent. If* $ReR=R$, *then*

(1) *For any right* R*-module* N, N *is an* FP*-injective right* R*-module if and only if* $N\otimes_R Re$ *is an* FP*-injective right* eRe*-module.*
(2) *For any right* eRe*-module* N, N *is an* FP*-injective right* eRe*-module if and only if* $N\otimes_{eRe} eR$ *is an* FP*-injective right* R*-module.*
(3) *For any right* R*-module* M, M *is an* FP*-projective right* R*-module if and only if* $M\otimes_R Re$ *is an* FP*-projective right* eRe*-module.*
(4) *For any right* eRe*-module* M, M *is an* FP*-projective right* eRe*-module if and only if* $M\otimes_{eRe} eR$ *is an* FP*-projective right* R*-module.*

Corollary 2.12 *Let* R *be ring and* $n\geq 1$ *be a positive integer. Then*

(1) *For any right* R*-module* N, N *is an* FP*-injective right* R*-module if and only if* $N\otimes_R M_n(R)e_{ii}$ *is an* FP*-injective right* $M_n(R)$*-module.*
(2) *For any right* $M_n(R)$*-module* N, N *is an* FP*-injective right* $M_n(R)$*-module if and only if* $N\otimes_{M_n(R)} e_{ii}M_n(R)$ *is an* FP*-injective right* R*-module.*
(3) *For any right* R*-module* M, M *is an* FP*-projective right* R*-module if and only if* $M\otimes_R M_n(R)e_{ii}$ *is an* FP*-projective right* $M_n(R)$*-module.*
(4) *For any right* $M_n(R)$*-module* M, M *is an* FP*-projective right* $M_n(R)$*-module if and only if* $M\otimes_{M_n(R)} e_{ii}M_n(R)$ *is an* FP*-projective right* R*-module. Where* e_{ii} *is the unit of Matrix, i=1,2,...n.*

Now let us discuss the properties of FP-projective modules and FP-injective modules under polynomial extensions. Let R be ring, $R[x]$ is polynomial ring over ring R. Suppose that M is a left R-module, set $M[x]=R[x]\otimes_R M$.

Theorem 2.13 *Let* R *be a commutative ring. Then the following are equivalent for any right* $R[x]$*-module* N:

(1) N *is an* FP*-injective right* R*-module.*
(2) $Hom_R(R[x],N)$ *is an* FP*-injective right* R*-module.*
(3) $Hom_R(R[x],N)$ *is an* FP*-injective right* $R[x]$*-module.*
(4) N *is an* FP*-injective right* $R[x]$*-module.*

Proof (1) $\Rightarrow$ (3) *Let* F *be a finitely presented right* $R[x]$*-module. Then* F *is a finitely presented right* R*-module. Since* $Ext^1_{R[x]}(F,Hom_R(R[x],N))\cong Ext^1_R$ $(F\otimes_{R[x]} R[x],N)\cong Ext^1_R(F,N)$, *we have* $Ext^1_R(F,N)=0$ *by (1). Therefore*

$Ext^1_{R[x]}(F, Hom_R(R[x], N)) = 0$, *and so* $Hom_R(R[x], N)$ *is an* FP*-injective right* $R[x]$*-module.*

(3) ⇒ (4) N *is an* FP*-injective right* $R[x]$*-module since* $N_{R[x]}$ *is isomorphic to a direct summand of* $Hom_R(R[x], N)$.

(4) ⇒*(1) If* F *be a finitely presented right* R*-module, then there is an exact sequence of right* R*-modules* $0 \to K \to P \to F \to 0$ *with* P *projective and both* K *and* P *finitely generated. Since* $R[x]$ *is flat* R*-module, we have the following right* $R[x]$*-module exact sequence* $0 \to K \otimes_R R[x] \to P \otimes_R R[x] \to F \otimes_R R[x] \to 0$. *Note that* $K \otimes_R R[x]$ *is a finitely generated right* $R[x]$*-module,* $P \otimes_R R[x]$ *is a finitely generated projective right* $R[x]$*-module, and so* $F \otimes_R R[x]$ *is a finitely presented right* $R[x]$*-module. Since* $Ext^1_R(F, N) \cong Ext^1_R(F, Hom_{R[x]}(R[x], N)) \cong Ext^1_{R[x]}(F \otimes_R R[x], M)$, *we have* $Ext^1_{R[x]}(F \otimes_R R[x], M) = 0$ *by (4). Thus* $Ext^1_R(F, N) = 0$.

(2) ⇔ (3) *is clear by the proof of* (1) ⇔ (3) ⇔ (4).

Theorem 2.14 *Let* R *be a commutative ring. Then the following are equivalent for any right* $R[x]$*-module* M:

(1) M *is an* FP*-projective right* R*-module.*
(2) $M \otimes_R R[x]$ *is an* FP*-projective right* R*-module.*
(3) $M \otimes_R R[x]$ *is an* FP*-projective right* $R[x]$*-module.*
(4) M *is an* FP*-projective right* $R[x]$*-module.*

Proof (1) ⇒ (3) *Suppose that* N *be an* FP*-injective right* $R[x]$*-module, we have* $Ext^1_R(M, N) = 0$ *since* N *is an* FP*-injective right* R*-module by Theorem 2.13. Note that* $Ext^1_{R[x]}(M \otimes_R R[x], N) \cong Ext^1_R(M, Hom_{R[x]}(R[x], N)) \cong Ext^1_R (M, N) = 0$ *by (1), we have* $Ext^1_{R[x]}(M \otimes_R R[x], N) = 0$, *and hence* $M \otimes_R R[x]$ *is an* FP*-projective right* $R[x]$*-module.*

(3) ⇒ (4) *Since* $M_{R[x]}$ *is isomorphic to a direct summand of* $M \otimes_R R[x]$, M *is an* FP*-projective right* $R[x]$*-module.*

(4) ⇒ (1) *Let* N *is an* FP*-injective right* R*-module. Then* $Hom_R(R[x], N)$ *is an* FP*-injective right* $R[x]$*-module by Theorem 2.12, and hence we get* $Ext^1_{R[x]}(M, Hom_R(R[x], N)) = 0$. *Note that* $Ext^1_R(M, N) \cong Ext^1_R(M \otimes_{R[x]} R[x], N) \cong Ext^1_{R[x]}(M, Hom_R(R[x], N))$, *therefore* $Ext^1_R(M, N) = 0$.

(2) ⟺ (3) *is clear by the proof of* (1) ⟺ (3) ⟺ (4).

Now let us discuss the properties of FP*-projective modules and* FP*-injective modules under localizations. Let* S *be a multipicatively closed subset of commutative ring* R *and* M *a* R*-module, then* $S^{-1}M \cong S^{-1}R \otimes_R M$.

Lemma 2.15 *Let* R *be a commutative ring,* S *is multiplicatively closed subset of* R*. If* $S^{-1}R$ *is a projective* R*-module, then any* FP*-injective* $S^{-1}R$*-module* N *is* FP*-injective* R*-module.*

Proof *Let* F *be a finitely presented right* R*-module. Then* $F \otimes_R S^{-1}R \cong S^{-1}F$

is a finitely presented right $S^{-1}R$-module by [1, Theorem 3.7.6], so $Ext^1_{S^{-1}R}(F\otimes_R S^{-1}R, N) = 0$. Since $S^{-1}R$ is a projective R-module, we have the following isomorphism $Ext^1_R(F,N) \cong Ext^1_R(F, Hom_{S^{-1}R}(S^{-1}R,N)) \cong Ext^1_{S^{-1}R}(F \otimes_R S^{-1}R, N)$ by [1, Exercise 9.21], and so it follows that $Ext^1_R(F,N) = 0$, and hence N is FP-injective R-module.

Theorem 2.16 *Let R be a commutative ring, S is multiplicatively closed subset of R. If $S^{-1}R$ is a projective R-module, then*

(1) For any FP-projective R-module M, $S^{-1}M$ is FP-projective $S^{-1}R$-module.
(2) For any FP-injective R-module N, $Hom_R(S^{-1}R,N)$ is FP-injective $S^{-1}R$-module.

Proof *(1) Suppose that M is an FP-projective R-module. Let N be an FP-injective $S^{-1}R$-module. Then N is an FP-injective R-module by Lemma 2.15, so $Ext^1_R(M,N) = 0$. Since $Ext^1_{S^{-1}R}(S^{-1}M,N) \cong Ext^1_{S^{-1}R}(M \otimes_R S^{-1}R, N)$ $\cong Ext^1_R(M, Hom_{S^{-1}R}(S^{-1}R,N)) \cong Ext^1_R(M,N)$, we have $Ext^1_{S^{-1}R}(S^{-1}M,N)$ $= 0$, and hence $S^{-1}M$ is FP-projective $S^{-1}R$-module.*
(2) Suppose that N is an FP-injective R-module. Let F be a finitely presented $S^{-1}R$-module. Then F is a finitely presented R-module, and so $Ext^1_R(F,N) = 0$. Note the isomorphism $Ext^1_{S^{-1}R}(F, Hom_R(S^{-1}R,N)) \cong Ext^1_R(S^{-1}R \otimes_{S^{-1}R} F, N) \cong Ext^1_R(F,N)$, thus we have $Ext^1_{S^{-1}R}(F, Hom_R(S^{-1}R,N)) = 0$, and hence $Hom_R(S^{-1}R,N)$ is FP-injective $S^{-1}R$-module.

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