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Some Properties of Vague Rings

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Abstract

In this work, using the definition of vague group and its some results, the concepts of vague ring and vague subring are defined, and the validity of some relevant classical results in these settings are investigated.

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1 Introduction

Fuzzy subgroups were introduced in [8] by Rosenfeld as a natural generalization of the concept of subgroup and have been widely studied. Following this, a new object related to groups called vague groups was introduced and studied in [2] by Demirci by forcing the operations of the group to be compatible with a given fuzzy equality. Later on, the general theory of vague algebraic notions was established in [4, 5, 6, 7] based on a fixed integral commutative, complete quasi-monoidal lattice (for short, cqm-lattice) $(L, \leq, \star)$.

This work introduces some elementary properties of vague ring and vague subring, and establishes some new results on the basis of the particular integral, commutative cqm-lattice $([0, 1], \leq, \wedge)$.

After this introductory Section, Section 2 is devoted to some definitions and properties related to vague groups and generalized vague subgroups that will be needed later. In Section 3, the definitions of vague ring and vague subring will be given and some basic properties of these concepts will be investigated.

2 Preliminaries

The notions of fuzzy equality, strong fuzzy function, vague group and generalized vague subgroup and their fundamental properties are introduced in

[1, 2, 4, 9, 10, 11]. Our aim in this section is to recall these notions and some of their elementary properties, which will be needed in this paper.

The symbols "$\wedge$" and "$\vee$" will always stand for the minimum and maximum operations between finitely many real numbers, respectively; and X, Y, G will always stand for crisp and nonempty sets in this paper.

Definition 2.1 *[1] A mapping $E_X : X \times X \to [0,1]$ is called a fuzzy equality on X if the following conditions are satisfied:*
(E.1) $E_X(x,y) = 1 \Longleftrightarrow x = y \ , \ \forall x, y \in X,$
(E.2) $E_X(x,y) = E_X(y,x) \ , \ \forall x, y \in X \ ,$
(E.3) $E_X(x,y) \wedge E_X(y,z) \leq E_X(x,z) \ , \ \forall x, y, z \in X.$

For $x, y \in X$, the real number $E_X(x,y)$ shows the degree of the equality of x and y. One can always define a fuzzy equality on X with respect to (abbreviated to "w.r.t.") the classical equality of the elements of X. Indeed, the mapping $E_X^c : X \times X \to [0,1]$, defined by

$$E_X^c(x,y) = \begin{cases} 1 & , \ if \ x = y \\ 0 & , \ otherwise \end{cases} \tag{1}$$

is obviously a fuzzy equality on X.

Definition 2.2 *[4] Let E_X and E_Y be two fuzzy equalities on X and Y, respectively. Then a fuzzy relation $\tilde{\circ}$ from X to Y (i.e., a fuzzy subset $\tilde{\circ}$ of $X \times Y$) is called a strong fuzzy function from X to Y w.r.t. the fuzzy equalities E_X and E_Y, denoted by $\tilde{\circ} : X \rightsquigarrow Y$, if the characteristic function $\mu_{\tilde{\circ}} : X \times Y \to [0,1]$ of $\tilde{\circ}$ satisfies the following two conditions:*
(F.1) For each $x \in X$, there exists $y \in Y$ such that $\mu_{\tilde{\circ}}(x,y) = 1$,
(F.2) For each $x_1, x_2 \in X$, $y_1, y_2 \in Y$,

$$\mu_{\tilde{\circ}}(x_1,y_1) \wedge \mu_{\tilde{\circ}}(x_2,y_2) \wedge E_X(x_1,x_2) \leq E_Y(y_1,y_2). \tag{2}$$

The concepts of vague binary operation on X and transitivity of a vague binary operation are defined as follows.

Definition 2.3 *[2, 4]*

(i) A strong fuzzy function $\tilde{\circ} : X \times X \rightsquigarrow X$ w.r.t. a fuzzy equality $E_{X\times X}$ on $X \times X$ and a fuzzy equality E_X on X is called a vague binary operation on X w.r.t. $E_{X\times X}$ and E_X. (For all $(x_1,x_2) \in X \times X$, $x_3 \in X$, $\mu_{\tilde{\circ}}((x_1,x_2),x_3)$ will be denoted by $\mu_{\tilde{\circ}}(x_1,x_2,x_3)$ for the sake of simplicity.)

(ii) A vague binary operation $\tilde{\circ}$ on X w.r.t. $E_{X\times X}$ and E_X is said to be transitive of the first order if $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(c,d) \leq \mu_{\tilde{\circ}}(a,b,d)$ for all $a, b, c, d \in X$.

(iii) *A vague binary operation* $\tilde{\circ}$ *on* X *w.r.t.* $E_{X\times X}$ *and* E_X *is said to be transitive of the second order if* $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(b,d) \leq \mu_{\tilde{\circ}}(a,d,c)$ *for all* $a,b,c,d \in X$.

(iv) *A vague binary operation* $\tilde{\circ}$ *on* X *w.r.t.* $E_{X\times X}$ *and* E_X *is said to be transitive of the third order if* $\mu_{\tilde{\circ}}(a,b,c) \wedge E_X(a,d) \leq \mu_{\tilde{\circ}}(d,b,c)$ *for all* $a,b,c,d \in X$.

Definition 2.4 *[2] Let* $\tilde{\circ}$ *be a vague binary operation on* G *w.r.t. a fuzzy equality* $E_{G\times G}$ *on* $G \times G$ *and a fuzzy equality* E_G *on* G. *Then*

(i) G *together with* $\tilde{\circ}$, *denoted by* $< G, \tilde{\circ}, E_{G\times G}, E_G >$ *or simply* $< G, \tilde{\circ} >$, *is called a vague semigroup if the characteristic function* $\mu_{\tilde{\circ}} : G \times G \times G \to [0,1]$ *of* $\tilde{\circ}$ *fulfills the condition: For all* $a,b,c,d,m,q,w \in G$,

$$\mu_{\tilde{\circ}}(b,c,d) \wedge \mu_{\tilde{\circ}}(a,d,m) \wedge \mu_{\tilde{\circ}}(a,b,q) \wedge \mu_{\tilde{\circ}}(q,c,w) \leq E_G(m,w). \tag{3}$$

(ii) *A vague semigroup* $< G, \tilde{\circ} >$ *is called a vague monoid if there exists a two-sided identity element* $e_{\tilde{\circ}} \in G$, *that is an element* $e_{\tilde{\circ}}$ *satisfying* $\mu_{\tilde{\circ}}(e_{\tilde{\circ}}, a, a) \wedge \mu_{\tilde{\circ}}(a, e_{\tilde{\circ}}, a) = 1$ *for each* $a \in G$.

(iii) *A vague monoid* $< G, \tilde{\circ} >$ *is called a vague group if for each* $a \in G$, *there exists a two-sided inverse element* $a^{-1} \in G$, *that is an element* a^{-1} *satisfying* $\mu_{\tilde{\circ}}(a^{-1}, a, e_{\tilde{\circ}}) \wedge \mu_{\tilde{\circ}}(a, a^{-1}, e_{\tilde{\circ}}) = 1$.

(iv) *A vague semigroup* $< G, \tilde{\circ} >$ *is said to be commutative (Abelian) if*

$$\mu_{\tilde{\circ}}(a,b,m) \wedge \mu_{\tilde{\circ}}(b,a,w) \leq E_G(m,w) \tag{4}$$

for each $a,b,m,w \in G$.

In the rest of this paper, the notation $< G, \tilde{\circ} >$ always stands for the vague group $< G, \tilde{\circ} >$ w.r.t. a fuzzy equality $E_{G\times G}$ on $G \times G$ and a fuzzy equality E_G on G.

Proposition 2.5 *[2] For a given vague group* $< G, \tilde{\circ} >$, *there exists a unique binary operation in the classical sense, denoted by* $\circ$, *on* G *such that* $< G, \circ >$ *is a group in the classical sense.*

The binary operation "$\circ$" in Proposition 2.5 is explicitly given by the equivalence

$$a \circ b = c \iff \mu_{\tilde{\circ}}(a,b,c) = 1, \quad \forall a,b,c \in G. \tag{5}$$

If $< G, \tilde{\circ} >$ is a vague group, then

$$\mu_{\tilde{\circ}}(a,b,a\circ b) = 1 \quad \text{and} \quad \mu_{\tilde{\circ}}(a,b,c) \leq E_G(a\circ b, c)\ ,\ \forall a,b,c \in G. \tag{6}$$

Theorem 2.6 *[2] Let $< G, \tilde{\circ} >$ be a vague group.*

(i) If the vague binary operation $\tilde{\circ}$ is transitive of the second order, then $E_G(a,b) = E_G(a^{-1}, b^{-1})$ for all $a, b \in G$.

(ii) $\mu_{\tilde{\circ}}(b^{-1}, a^{-1}, u) \wedge \mu_{\tilde{\circ}}(a,b,v) \leq E_G(u, v^{-1}) \wedge E_G(v, u^{-1})$ for all $a, b, u, v \in G$.

For a given fuzzy equality E_G on G and for a crisp subset A of G, the restriction of the mapping E_G to $A \times A$, denoted by E_A, is obviously a fuzzy equality on A.

Definition 2.7 *[10] Let $< G, \tilde{\circ} >$ be a vague group and A be a nonempty, crisp subset of G. Let $\tilde{\odot}$ be a vague binary operation on A such that*

$$\mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c), \ \forall a,b,c \in A. \tag{7}$$

If $< A, \tilde{\odot} >$ is itself a vague group w.r.t. the fuzzy equalities $E_{A\times A}$ on $A \times A$ and E_A on A, then $< A, \tilde{\odot} >$ is said to be a generalized vague subgroup of $< G, \tilde{\circ} >$, denoted by $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.

For a given vague group $< G, \tilde{\circ} >$, because of the uniqueness of the identity and the inverse of an element of $< G, \tilde{\circ} >$, it can be easily seen that if $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$, then the identity of $< A, \tilde{\odot} >$ and the inverse of $x \in A$ w.r.t. $< A, \tilde{\odot} >$ are the identity of $< G, \tilde{\circ} >$ and the inverse of $x \in A$ w.r.t. $< G, \tilde{\circ} >$, i.e., $e_A = e_G$ and $x_A^{-1} = x_G^{-1}$, respectively.

Proposition 2.8 *[10] Let $< G, \tilde{\circ} >$ be a vague group. If $< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$ and $< B, \tilde{\bullet} > \overset{\text{v.s}}{\leq} < A, \tilde{\odot} >$, then $< B, \tilde{\bullet} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.*

Proposition 2.9 *[10] Let $< G, \tilde{\circ} >$ be a vague group, $\emptyset \neq A \subseteq G$ and let $\tilde{\odot}$ be a vague binary operation on A. Then*

$$< A, \tilde{\odot} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} > \Leftrightarrow \begin{array}{ll} (i) & \text{For each } x \in A, \ x^{-1} \in A, \text{ and} \\ (ii) & \mu_{\tilde{\odot}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c), \ \forall a,b,c \in A. \end{array} \tag{8}$$

Corollary 2.10 *[10] Let $< G, \tilde{\circ} >$ be a vague group and $\tilde{\bullet}$ be a vague binary operation on G such that $\mu_{\tilde{\bullet}}(a,b,c) \leq \mu_{\tilde{\circ}}(a,b,c)$ for all $a,b,c \in G$. Let $e_{\tilde{\circ}}$ be an identity element of G. Then, $< \{e_{\tilde{\circ}}\}, \tilde{\circ} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$ and $< G, \tilde{\bullet} > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$.*

Corollary 2.11 *[10] Let $< G, \tilde{\circ} >$ be a vague group, and let $< A_j, \tilde{\bullet}_j > \overset{\text{v.s}}{\leq} < G, \tilde{\circ} >$ for all $j \in J$. If $\tilde{\star}$ is a vague binary operation on $\bigcap_{j\in J} A_j$ such that*

$$\mu_{\tilde{\star}}(x,y,z) \leq \bigwedge_{j\in J} \mu_{\tilde{\bullet}_j}(x,y,z), \ \forall x,y,z \in \bigcap_{j\in J} A_j \ , \tag{9}$$

then $< \bigcap_{j\in J} A_j, \tilde{\star} > \overset{\text{v.s}}{\leq} < A_j, \tilde{\bullet}_j >$.

3 Vague Rings

In a similar fashion to classical algebra, the notion of vague ring can be given in the following way:

Definition 3.1 *Let $E_{\mathcal{H}\times\mathcal{H}}$ and $E_{\mathcal{H}}$ be fuzzy equalities on $\mathcal{H}\times\mathcal{H}$ and $\mathcal{H}$, respectively. Let $\tilde{\circ}$, $\tilde{\bullet}$ be two vague binary operations on $\mathcal{H}$. Then, the 3-tuple $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ is called a vague ring w.r.t. $E_{\mathcal{H}\times\mathcal{H}}$ and $E_{\mathcal{H}}$ if the following three conditions are satisfied:*

(VR.1) $<\mathcal{H},\tilde{\circ}>$ *is a commutative vague group,*

(VR.2) $<\mathcal{H},\tilde{\bullet}>$ *is a vague semigroup,*

(VR.3) $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *satisfies distributive laws, i.e.,* $\forall a,b,c,d,t,x,y,z\in\mathcal{H}$,

$$\mu_{\tilde{\bullet}}(x,y,a)\wedge\mu_{\tilde{\bullet}}(x,z,b)\wedge\mu_{\tilde{\circ}}(a,b,c)\wedge\mu_{\tilde{\circ}}(y,z,d)\wedge\mu_{\tilde{\bullet}}(x,d,t)\leq E_{\mathcal{H}}(t,c), \tag{10}$$

$$\mu_{\tilde{\bullet}}(x,z,a)\wedge\mu_{\tilde{\bullet}}(y,z,b)\wedge\mu_{\tilde{\circ}}(a,b,c)\wedge\mu_{\tilde{\circ}}(x,y,d)\wedge\mu_{\tilde{\bullet}}(d,z,t)\leq E_{\mathcal{H}}(t,c). \tag{11}$$

(VR.4) A vague ring $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ is said to be a vague ring with identity if there exists $e_{\tilde{\bullet}}\in\mathcal{H}$ such that $\mu_{\tilde{\bullet}}(x,e_{\tilde{\bullet}},x)\wedge\mu_{\tilde{\bullet}}(e_{\tilde{\bullet}},x,x)=1$ for each $x\in\mathcal{H}$.

(VR.5) A vague ring $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ is said to be a commutative (Abelian) if

$$\mu_{\tilde{\bullet}}(x,y,s)\wedge\mu_{\tilde{\bullet}}(y,x,t)\leq E_{\mathcal{H}}(s,t)\ ,\ \forall x,y,s,t\in\mathcal{H}. \tag{12}$$

In this work, since the particular integral, commutative cqm-lattice is being studied $([0,1],\leq,\wedge)$, Definition 3.1 corresponds to a special case of Definition 7.12 in [4].

In the rest of this paper, the notation $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ always stands for the vague ring $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ w.r.t. $E_{\mathcal{H}\times\mathcal{H}}$ and $E_{\mathcal{H}}$. If $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ is a vague ring, then we denote the inverse of a by $-a$ w.r.t. the vague group $<\mathcal{H},\tilde{\circ}>$; additionally if $<\mathcal{H},\tilde{\bullet}>$ is a vague group, then we denote the inverse of a by a^{-1} w.r.t. the vague group $<\mathcal{H},\tilde{\bullet}>$.

Example 3.2 *Let $<\mathcal{H},\circ,\bullet>$ be a ring. For $x,y,a,b\in\mathcal{H}$ and $\alpha,\beta,\gamma,\nu\in\mathbb{R}$ such that $0\leq\nu\leq\gamma\leq\beta\leq\alpha<1$, considering the fuzzy equalities*

$$E_{\mathcal{H}}(a,b):=\begin{cases}1 & ,\quad a=b\\ \alpha & ,\quad otherwise\end{cases} \tag{13}$$

on $\mathcal{H}$ *and*

$$E_{\mathcal{H}\times\mathcal{H}}((a,b),(x,y)) := \begin{cases} 1 & , \quad (a,b)=(x,y) \\ \beta & , \quad otherwise \end{cases} \tag{14}$$

on $\mathcal{H}\times\mathcal{H}$. *And, considering the vague binary operations*

$$\tilde{\circ} : \mathcal{H}\times\mathcal{H} \rightsquigarrow \mathcal{H} \ , \ \mu_{\tilde{\circ}}(a,b,c) := \begin{cases} 1 & , \quad a\circ b = c \\ \gamma & , \quad otherwise \end{cases} \tag{15}$$

and

$$\tilde{\bullet} : \mathcal{H}\times\mathcal{H} \rightsquigarrow \mathcal{H} \ , \ \mu_{\tilde{\bullet}}(a,b,c) := \begin{cases} 1 & , \quad a\bullet b = c \\ \nu & , \quad otherwise. \end{cases} \tag{16}$$

In this case, it is clearly seen that $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *is a vague ring from the inequality in (6) and the condition* $(E.3)$

Proposition 3.3 *[4] If* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *is a vague ring, then* $<\mathcal{H},\circ,\bullet>$ *is a ring.*

Proposition 3.4 *Let* $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ *be a vague ring and* $e_{\tilde{\circ}}$ *an identity element of the vague group* $<\mathcal{H},\tilde{\circ}>$. *Then, the following statements are satisfied for all* $m,n,t,u,v,w,x,y,z\in\mathcal{H}$.

(1) $\mu_{\tilde{\bullet}}(x,e_{\tilde{\circ}},m)\wedge\mu_{\tilde{\bullet}}(e_{\tilde{\circ}},x,n)\le E_{\mathcal{H}}(m,n)$.

(2) $\mu_{\tilde{\bullet}}(-x,-y,m)\wedge\mu_{\tilde{\bullet}}(x,y,n)\le E_{\mathcal{H}}(m,n)$.

(3) *If the vague binary operation* $\tilde{\circ}$ *is transitive of the second order, then* $\mu_{\tilde{\bullet}}(x,-y,m)\wedge\mu_{\tilde{\bullet}}(x,y,n)\le E_{\mathcal{H}}(m,-n)$ *and* $\mu_{\tilde{\bullet}}(-x,y,m)\wedge\mu_{\tilde{\bullet}}(x,y,n)\le E_{\mathcal{H}}(m,-n)$.

(4) *Let the vague binary operation* $\tilde{\circ}$ *be transitive of the second and third orders.*

(i) *If the vague binary operation* $\tilde{\bullet}$ *is transitive of the second order, then*

$$\mu_{\tilde{\circ}}(y,-z,u)\wedge\mu_{\tilde{\bullet}}(x,u,m)\wedge\mu_{\tilde{\bullet}}(x,y,v)\wedge\mu_{\tilde{\bullet}}(x,z,t)\wedge\mu_{\tilde{\circ}}(v,-t,n)\le E_{\mathcal{H}}(m,n). \tag{17}$$

(ii) *If the vague binary operation* $\tilde{\bullet}$ *is transitive of the third order, then*

$$\mu_{\tilde{\circ}}(x,-y,u)\wedge\mu_{\tilde{\bullet}}(u,z,m)\wedge\mu_{\tilde{\bullet}}(x,z,v)\wedge\mu_{\tilde{\bullet}}(y,z,t)\wedge\mu_{\tilde{\circ}}(v,-t,n)\le E_{\mathcal{H}}(m,n). \tag{18}$$

Proof (1): Since $<\mathcal{H},\tilde{\circ},\tilde{\bullet}>$ is a vague ring then $<\mathcal{H},\circ,\bullet>$ is a ring from Proposition 3.3. Thus, $e_{\tilde{\circ}}\bullet x = x\bullet e_{\tilde{\circ}} = e_{\tilde{\circ}}$ for all $x\in\mathcal{H}$. So, using the inequality in (6) and the condition $(E.3)$, we can write

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(x,e_{\tilde{\circ}},m)\wedge\mu_{\tilde{\bullet}}(e_{\tilde{\circ}},x,n) & \leq & E_{\mathcal{H}}(x\bullet e_{\tilde{\circ}},m)\wedge E_{\mathcal{H}}(e_{\tilde{\circ}}\bullet x,n) \\ & \leq & E_{\mathcal{H}}(e_{\tilde{\circ}},m)\wedge E_{\mathcal{H}}(e_{\tilde{\circ}},n) \\ & \leq & E_{\mathcal{H}}(m,n)\ ,\ \forall m,n\in\mathcal{H}. \end{array} \tag{19}$$

(2): By using the inequality in (6) and the condition $(E.3)$, we get the following inequalities

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(-x,-y,m)\wedge\mu_{\tilde{\bullet}}(x,y,n) & \leq & E_{\mathcal{H}}((-x)\bullet(-y),m)\wedge E_{\mathcal{H}}(x\bullet y,n) \\ & = & E_{\mathcal{H}}(x\bullet y,m)\wedge E_{\mathcal{H}}(x\bullet y,n) \\ & \leq & E_{\mathcal{H}}(m,n)\ ,\ \forall x,y,m,n\in\mathcal{H}. \end{array} \tag{20}$$

(3): We suppose that the vague binary operation $\tilde{\circ}$ is transitive of the second order. In this case, we have $E_{\mathcal{H}}(x\bullet y,n) = E_{\mathcal{H}}(-(x\bullet y),-n)$ from Theorem 2.6, so we can write the following inequalities from the inequality in (6) and the condition $(E.3)$,

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(x,-y,m)\wedge\mu_{\tilde{\bullet}}(x,y,n) & \leq & E_{\mathcal{H}}(x\bullet(-y),m)\wedge E_{\mathcal{H}}(x\bullet y,n) \\ & = & E_{\mathcal{H}}(-(x\bullet y),m)\wedge E_{\mathcal{H}}(-(x\bullet y),-n) \\ & \leq & E_{\mathcal{H}}(m,-n) \end{array} \tag{21}$$

and

$$\begin{array}{rcl} \mu_{\tilde{\bullet}}(-x,y,m)\wedge\mu_{\tilde{\bullet}}(x,y,n) & \leq & E_{\mathcal{H}}((-x)\bullet y,m)\wedge E_{\mathcal{H}}(x\bullet y,n) \\ & = & E_{\mathcal{H}}(-(x\bullet y),m)\wedge E_{\mathcal{H}}(-(x\bullet y),-n) \\ & \leq & E_{\mathcal{H}}(m,-n). \end{array} \tag{22}$$

(4): (i) From classical algebra, we know that $E_{\mathcal{H}}(x\bullet(y\circ(-z)),m) = E_{\mathcal{H}}((x\bullet y)\circ(-(x\bullet z)),m)$ for all $x,y,z,m\in\mathcal{H}$. Since the vague binary operation $\tilde{\circ}$ is transitive of the second order, by making use of Theorem 2.6, we get $E_{\mathcal{H}}(x\bullet z,t) = E_{\mathcal{H}}(-(x\bullet z),-t)$ for all $x,z,t\in\mathcal{H}$.

If we denote

$$\alpha = \mu_{\tilde{\circ}}(y,-z,u)\wedge\mu_{\tilde{\bullet}}(x,u,m)\wedge\mu_{\tilde{\bullet}}(x,y,v)\wedge\mu_{\tilde{\bullet}}(x,z,t)\wedge\mu_{\tilde{\circ}}(v,-t,n), \tag{23}$$

then we get the following inequalities by using the vague binary operation $\tilde{\circ}$ is transitive of the second and third orders, the vague binary operation $\tilde{\bullet}$ is transitive of the second order, the inequality in (6) and the condition $(E.3)$.

$$\begin{array}{rcl} \alpha & \leq & E_{\mathcal{H}}(y\circ(-z),u)\wedge\mu_{\tilde{\bullet}}(x,u,m)\wedge E_{\mathcal{H}}(x\bullet y,v)\wedge E_{\mathcal{H}}(x\bullet z,t)\wedge\mu_{\tilde{\circ}}(v,-t,n) \\ & \leq & \mu_{\tilde{\bullet}}(x,y\circ(-z),m)\wedge E_{\mathcal{H}}(x\bullet y,v)\wedge E_{\mathcal{H}}(x\bullet z,t)\wedge\mu_{\tilde{\circ}}(v,-t,n) \\ & \leq & \mu_{\tilde{\bullet}}(x,y\circ(-z),m)\wedge E_{\mathcal{H}}(x\bullet y,v)\wedge E_{\mathcal{H}}(-(x\bullet z),-t)\wedge\mu_{\tilde{\circ}}(v,-t,n) \\ & \leq & E_{\mathcal{H}}(x\bullet(y\circ(-z)),m)\wedge E_{\mathcal{H}}(x\bullet y,v)\wedge\mu_{\tilde{\circ}}(v,-(x\bullet z),n) \\ & \leq & E_{\mathcal{H}}(x\bullet(y\circ(-z)),m)\wedge\mu_{\tilde{\circ}}(x\bullet y,-(x\bullet z),n) \\ & \leq & E_{\mathcal{H}}(x\bullet(y\circ(-z)),m)\wedge E_{\mathcal{H}}((x\bullet y)\circ(-(x\bullet z)),n) \\ & \leq & E_{\mathcal{H}}(m,n). \end{array} \tag{24}$$

(ii) Since the vague binary operation $\tilde{\circ}$ is transitive of the second order, we have $E_{\mathcal{H}}(y \bullet z, t) = E_{\mathcal{H}}(-(y \bullet z), -t)$ for all $x, y, z, t \in \mathcal{H}$ from Theorem 2.6. Furthermore, because $< \mathcal{H}, \circ, \bullet >$ is a ring, we can write that $E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) = E_{\mathcal{H}}((x \bullet z) \circ (-(y \bullet z)), m)$ for all $x, y, z, m \in \mathcal{H}$. If we denote

$$\beta = \mu_{\tilde{\circ}}(x, -y, u) \wedge \mu_{\tilde{\bullet}}(u, z, m) \wedge \mu_{\tilde{\bullet}}(x, z, v) \wedge \mu_{\tilde{\bullet}}(y, z, t) \wedge \mu_{\tilde{\circ}}(v, -t, n), \quad (25)$$

then we can write the following inequalities from the hypothesis, the inequality in (6) and the condition $(E.3)$.

$$\begin{aligned}
\beta &\leq E_{\mathcal{H}}(x \circ (-y), u) \wedge \mu_{\tilde{\bullet}}(u, z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge E_{\mathcal{H}}(y \bullet z, t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \\
&\leq E_{\mathcal{H}}(x \circ (-y), u) \wedge \mu_{\tilde{\bullet}}(u, z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge E_{\mathcal{H}}(-(y \bullet z), -t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \\
&\leq \mu_{\tilde{\bullet}}((x \circ (-y)), z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge E_{\mathcal{H}}(-(y \bullet z), -t) \wedge \mu_{\tilde{\circ}}(v, -t, n) \\
&\leq E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) \wedge E_{\mathcal{H}}(x \bullet z, v) \wedge \mu_{\tilde{\circ}}(v, -(y \bullet z), n) \\
&\leq E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) \wedge \mu_{\tilde{\circ}}(x \bullet z, -(y \bullet z), n) \\
&\leq E_{\mathcal{H}}((x \circ (-y)) \bullet z, m) \wedge E_{\mathcal{H}}((x \bullet z) \circ (-(y \bullet z)), n) \\
&\leq E_{\mathcal{H}}(m, n).
\end{aligned} \quad (26)$$

Definition 3.5 *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring and A be a nonempty, crisp subset of $\mathcal{H}$. Let $\tilde{\oplus}$ and $\tilde{\odot}$ be two vague binary operations on A such that*

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \ , \quad \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \quad \forall a, b, c \in A. \quad (27)$$

If $< A, \tilde{\oplus}, \tilde{\odot} >$ is itself a vague ring w.r.t. $E_{A \times A}$ and E_A, then $< A, \tilde{\oplus}, \tilde{\odot} >$ is said to be a vague subring of $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$, denoted by $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$.

The following propositions and corollaries state that some results of classical algebra are also valid for vague algebra.

Proposition 3.6 *Let $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ be a vague ring and $A \subseteq \mathcal{H}$. Let $\tilde{\oplus}$ and $\tilde{\odot}$ be two vague binary operations on A. Then the following equivalence is satisfied:*

$$< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} > \ \Leftrightarrow \ \begin{array}{ll} (i) & < A, \tilde{\oplus} > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} > \\ (ii) & \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \forall a, b, c \in A. \end{array} \quad (28)$$

Proof. $(\Rightarrow)$: Obvious from Definition 3.5.

$(\Leftarrow)$**:** By making use of (i) and (ii), we can write

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \mu_{\tilde{\circ}}(a, b, c) \text{ and } \mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c), \ \forall a, b, c \in A. \quad (29)$$

Therefore, it is sufficient to show that $< A, \tilde{\oplus}, \tilde{\odot} >$ is a vague ring. The conditions $(VR.1)$ and $(VR.2)$ are satisfied for $< A, \tilde{\oplus}, \tilde{\odot} >$ under the assumptions (i) and (ii). On the other hand, since $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ satisfies distributive laws, the condition $(VR.3)$ is also obtained for $< A, \tilde{\oplus}, \tilde{\odot} >$. Hence, $< A, \tilde{\oplus}, \tilde{\odot} >$ must be a vague ring, i.e., $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$.

The following corollary explains that the intersection of vague subrings is also a vague subring.

Corollary 3.7 *Let* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *be a vague ring and* $< A_j, \tilde{\oplus}_j, \tilde{\odot}_j > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *for all* $j \in J = \{1, 2, ..., n\}$. *Let* $A = \bigcap_{j\in J} A_j$, *and let* $\tilde{\oplus}$, $\tilde{\odot}$ *be two vague binary operations on* A *such that*

$$\mu_{\tilde{\oplus}}(a, b, c) \leq \bigwedge_{j\in J} \mu_{\tilde{\oplus}_j}(a, b, c) \quad \text{and} \quad \mu_{\tilde{\odot}}(a, b, c) \leq \bigwedge_{j\in J} \mu_{\tilde{\odot}_j}(a, b, c) \quad \forall a, b, c \in A. \tag{30}$$

Then, $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$.

Proof. Because of $< A_j, \tilde{\oplus}_j, \tilde{\odot}_j > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$, we have $< A_j, \tilde{\oplus}_j > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} >$ for all $j \in J$. Thus, it is clearly seen that, $< A, \tilde{\oplus} > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} >$ from Corollary 2.11 and Proposition 2.8. On the other hand, since $\mu_{\tilde{\odot}}(a, b, c) \leq \mu_{\tilde{\bullet}}(a, b, c)$ for all $a, b, c \in A$, we obtain $< A, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ from Proposition 3.6.

Corollary 3.8 *Let* $< \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *be a vague ring and* $e_{\tilde{\circ}}$ *be an identity element of* $< \mathcal{H}, \tilde{\circ} >$. *Let* $\tilde{\oplus}$ *and* $\tilde{\odot}$ *be two vague binary operations on* $\mathcal{H}$ *such that* $\mu_{\tilde{\oplus}}(x, y, z) \leq \mu_{\tilde{\circ}}(x, y, z)$, $\mu_{\tilde{\odot}}(x, y, z) \leq \mu_{\tilde{\bullet}}(x, y, z)$ *for all* $x, y, z \in \mathcal{H}$. *Then the following properties are satisfied:*

(a) $< \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ *(b)* $< \mathcal{H}, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$

Proof. We know that $< \{e_{\tilde{\circ}}\}, \tilde{\circ} > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} >$ and $< \mathcal{H}, \tilde{\oplus} > \overset{\text{v.s}}{\leq} < \mathcal{H}, \tilde{\circ} >$ from Corollary 2.10. Using the inequalities in hypothesis and Proposion 3.6, we have $< \{e_{\tilde{\circ}}\}, \tilde{\circ}, \tilde{\bullet} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$ and $< \mathcal{H}, \tilde{\oplus}, \tilde{\odot} > \overset{v.r}{\leq} < \mathcal{H}, \tilde{\circ}, \tilde{\bullet} >$. This completes the proof.

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