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On Intuitionistic Fuzzy H-Ideals in BCK-Algebras

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Abstract

In this paper, we introduce the concept of intuitionistic fuzzy H-ideals in BCK-algebras and investigate some of its properties.

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1 Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [9] in 1965, several researchers explored on the generalization of the notion of fuzzy sets. The concept of intuitionistic fuzzy sets was introduced by Atanassov [1], as a generalization of the notion of fuzzy sets. In [5], Zhan and Tan introduced the fuzzy H-ideals in BCK-algebras. In this paper, we introduce the concept of intuitionisitc fuzzy H-ideals in BCK-algebras and obtaine some results.
A BCK-algebra is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the following axioms:
(1). $(x * y) * (x * z) \leq (z * y)$,
(2). $x * (x * y) \leq y$,
(3). $x \leq x$,
(4). $x \leq y$, $y \leq x \Rightarrow x = y$,
(5). $0 \leq x$, where $x \leq y$ is defined by $x * y = 0$.

Definition 1.1 *A subset I of a BCK-algebra $(X, *, 0)$ is called an ideal of X, if for any $x, y \in X$*

1. $0 \in I$,
2. $x * y$ *and* $y \in I \Rightarrow x \in I$.

Definition 1.2 *An ideal* I *of a BCK-algebra* $(X, *, 0)$ *is called closed if* $0 * x \in I$, *for all* $x \in I$.

Definition 1.3 *An intuitionistic fuzzy set* A *in a non-empty set* X *is an object having the form* $A = \{(x, \mu_A(x), \lambda_A(x))/x \in X\}$, *where the function* $\mu_A : X \to [0, 1]$ *and* $\lambda_A : X \to [0, 1]$ *denoted the degree of membership (namely* $\mu_A(x)$*) and the degree of non membership (namely* $\lambda_A(x)$*) of each element* $x \in X$ *to the set* A *respectively, and* $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ *for all* $x \in X$.

Definition 1.4 *An intuitionistic fuzzy set* $A = (X, \mu_A, \lambda_A)$ *in* X *is called an intuitionistic fuzzy ideal of* X, *if it satisfies the following axioms:*
(IF1) $\mu_A(0) \geq \mu_A(x)$ *and* $\lambda_A(0) \leq \lambda_A(x)$,
(IF2) $\mu_A(x) \geq min\{\mu_A(x * y), \mu_A(y)\}$,
(IF3) $\lambda_A(x) \leq max\{\lambda_A(x * y), \lambda_A(y)\}$, *for all* $x, y \in X$.

Definition 1.5 *An intuitionistic fuzzy set* $A = (X, \mu_A, \lambda_A)$ *in* X *is called an intuitionistic fuzzy closed ideal of X, if it satisfies (IF2),(IF3) and the following:*
(IF4) $\mu_A(0 * x) \geq \mu_A(x)$ *and* $\lambda_A(0 * x) \leq \lambda_A(x)$, *for all* $x \in X$.

Definition 1.6 *A non-empty subset* I *of BCK-algebra* X *is called an H-ideal of* X, *if*

1. $0 \in I$,

2. $x * (y * z) \in I$ *and* $y \in I \Rightarrow x * z \in I$.

Definition 1.7 *A fuzzy subset* μ *in a BCK-algebra* X *is called a fuzzy H-ideal of* X, *if*

1. $\mu(0) \geq \mu(x)$,

2. $\mu(x * z) \geq min\{\mu(x * (y * z)), \mu(y)\}$, *for all* $x, y, z \in X$.

Definition 1.8 *Let* μ *be a fuzzy set in* X. *The complement of* μ *is denoted by* $\bar{\mu}$ *and is defined as* $\bar{\mu}(x) = 1 - \mu(x)$, *for all* $x \in X$.

Definition 1.9 *Let* $A = (X, \mu_A, \lambda_A)$ *be an intuitionistic fuzzy set in* X. *Then (i).* $\neg A = (X, \mu_A, \bar{\mu_A})$ *and (ii).* $\diamond A = (X, \bar{\lambda_A}, \lambda_A)$.

2 Intuitionistic Fuzzy H-ideals

Definition 2.1 *An intuitionistic fuzzy set $A = (X, \mu_A, \lambda_A)$ in a BCK-algebra X is called an intuitionistic fuzzy H-ideal of X, if*
(IFH 1) $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$,
(IFH 2) $\mu_A(x * z) \geq min\{\mu_A(x * (y * z)), \mu_A(y)\}$,
(IFH 3) $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\}$, for all $x, y, z \in X$.

Definition 2.2 *An intuitionistic fuzzy set $A = (X, \mu_A, \lambda_A)$ in a BCK-algebra X is called an intuitionistic fuzzy closed H-ideal of X, if it satisfies (IFH 2),(IFH 3) and the following:*
(IFH 4) $\mu_A(0 * x) \geq \mu_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$, for all $x \in X$.

Definition 2.3 *Let $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy set in a BCK-algebra X. The set $U(\mu_A; s) = \{x \in X/\mu_A(x) \geq s\}$ is called upper s-level of μ_A and the set $L(\lambda_A; t) = \{x \in X/\lambda_A(x) \leq t\}$ is called lower t-level of λ_A.*

Lemma 2.4 *If $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy H-ideal of a BCK-algebra X, then we have the following $x \leq a \Rightarrow \mu_A(x) \geq \mu_A(a)$ and $\lambda_A(x) \leq \lambda_A(a)$, for all $x, a \in X$.*

Proof. Let $x, a \in X$ such that $x \leq a \Rightarrow x * a = 0$. Consider $\mu_A(x) = \mu_A(x * 0) \geq min\{\mu_A(x * (a * 0)), \mu_A(a)\} = min\{\mu_A(x * a), \mu_A(a)\} = \mu_A(a)$ and $\lambda_A(x) = \lambda_A(x * 0) \leq max\{\lambda_A(x * (a * 0)), \lambda_A(a)\} = max\{\lambda_A(x * a), \lambda_A(a)\} = \lambda_A(a)$.

Theorem 2.5 *Let $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy H-ideal of a BCK-algebra X. Then so is $\neg A = (X, \mu_A, \bar{\mu_A})$.*

Proof. We have

$$\mu_A(0) \geq \mu_A(x) \Rightarrow 1 - \bar{\mu}_A(0) \geq 1 - \bar{\mu}_A(x) \Rightarrow \bar{\mu}_A(0) \leq \bar{\mu}_A(x),$$

for any $x \in X$. Consider, for any $x, y, z \in X$,

$$\begin{aligned}
&\mu_A(x * z) \geq \min\{\mu_A(x * (y * z)), \mu_A(y)\} \\
&\Rightarrow 1 - \bar{\mu}_A(x * z) \geq \min\{1 - \bar{\mu}_A(x * (y * z)), 1 - \bar{\mu}_A(y)\} \\
&\Rightarrow \bar{\mu}_A(x * z) \leq 1 - \min\{1 - \bar{\mu}_A(x * (y * z)), 1 - \bar{\mu}_A(y)\} \\
&\Rightarrow \bar{\mu}_A(x * z) \leq \max\{\bar{\mu}_A(x * (y * z)), \bar{\mu}_A(y)\}.
\end{aligned}$$

Hence $\neg A = (X, \mu_A, \bar{\mu_A})$ is an IFH-ideal of X.

Theorem 2.6 *Let $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy H-ideal of a BCK-algebra X. Then so is $\diamond A = (X, \bar{\lambda}_A, \lambda_A)$.*

Proof. We have

$$\lambda_A(0) \leq \lambda_A(x) \Rightarrow 1 - \bar{\lambda}_A(0) \leq 1 - \bar{\lambda}_A(x) \Rightarrow \bar{\lambda}_A(0) \geq \bar{\lambda}_A(x),$$

for any $x \in X$. Consider, for any $x, y, z \in X$,

$$\begin{aligned} &\lambda_A(x * z) \leq \max\{\lambda_A(x * (y * z)), \lambda_A(y)\} \\ &\Rightarrow 1 - \bar{\lambda}_A(x * z) \leq \max\{1 - \bar{\lambda}_A(x * (y * z)), 1 - \bar{\lambda}_A(y)\} \\ &\Rightarrow \bar{\lambda}_A(x * z) \geq 1 - \max\{1 - \bar{\lambda}_A(x * (y * z)), 1 - \bar{\lambda}_A(y)\} \\ &\Rightarrow \bar{\lambda}_A(x * z) \geq \min\{\bar{\lambda}_A(x * (y * z)), \bar{\lambda}_A(y)\}. \end{aligned}$$

Hence $\diamond A = (X, \bar{\lambda_A}, \lambda_A)$ is an IFH-ideal of X.

Theorem 2.7 *$A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy H-ideal of a BCK-algebra X if and only if $\neg A = (X, \mu_A, \bar{\mu_A})$ and $\diamond A = (X, \bar{\lambda_A}, \lambda_A)$ are intuitionistic fuzzy H-ideals of a BCK-algebra X.*

Theorem 2.8 *If $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy closed H-ideal of a BCK-algebra X, then so is $\neg A = (X, \mu_A, \bar{\mu_A})$.*

Proof. For any $x \in X$, we have

$$\mu_A(0 * x) \geq \mu_A(x) \Rightarrow 1 - \bar{\mu}_A(0 * x) \geq 1 - \bar{\mu}_A(x) \Rightarrow \bar{\mu}_A(0 * x) \leq \bar{\mu}_A(x).$$

Hence $\neg A = (X, \mu_A, \bar{\mu_A})$ is closed H-ideal of X.

Theorem 2.9 *If $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy closed H-ideal of a BCK-algebra X, then so is $\diamond A = (X, \bar{\lambda_A}, \lambda_A)$.*

Proof. For any $x \in X$, We have

$$\lambda_A(0 * x) \leq \lambda_A(x) \Rightarrow 1 - \bar{\lambda}_A(0 * x) \leq 1 - \bar{\lambda}_A(x) \Rightarrow \bar{\lambda}_A(0 * x) \geq \bar{\lambda}_A(x).$$

Hence, $\diamond A = (X, \bar{\lambda_A}, \lambda_A)$ is an intuitionistic fuzzy closed H-ideal of X.

Theorem 2.10 *$A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy closed H-ideal of a BCK-algebra X if and only if $\neg A = (X, \mu_A, \bar{\mu_A})$ and $\diamond A = (X, \bar{\lambda_A}, \lambda_A)$ are intuitionistic fuzzy closed H-ideals of BCK-algebra X.*

Theorem 2.11 *$A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy H-ideal of a BCK-algebra X if and only if the non-empty upper s-level cut $U(\mu_A; s)$ and the non-empty lower t-level cut $L(\lambda_A; t)$ are H-ideals of X, for any $s, t \in [0, 1]$.*

Proof. Suppose $A = (X, \mu_A, \lambda_A)$ is an IFH-ideal of a BCK-algebra X. For any $s, t \in [0, 1]$, define the sets $U(\mu_A; s) = \{x \in X/\mu_A(x) \geq s\}$ and $L(\lambda_A; t) = \{x \in X/\lambda_A(x) \leq t\}$. Since $L(\lambda_A; t) \neq \phi$, for $x \in L(\lambda_A; t) \Rightarrow \lambda_A(x) \leq t \Rightarrow \lambda_A(0) \leq t \Rightarrow 0 \in L(\lambda_A; t)$. Let $x * (y * z) \in L(\lambda_A; t)$ and $y \in L(\lambda_A; t)$ implies $\lambda_A(x*(y*z)) \leq t$ and $\lambda_A(y) \leq t$. Since, for all $x, y, z \in X$, $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\} \leq max\{t, t\} = t \Rightarrow \lambda_A(x * z) \leq t$. Therefore $x * z \in L(\lambda_A; t)$, for all $x, y, z \in X$. Hence $L(\lambda_A; t)$ is an H-ideal of X. Similarly, we can prove $U(\mu_A; s)$ is an H-ideal of X.
Conversly, suppose that $U(\mu_A; s)$ and $L(\lambda_A; t)$ are H-ideal of X, for any $s, t \in [0, 1]$. If possible, assume $x_o, y_o \in X$ such that $\mu_A(0) < \mu_A(x_o)$ and $\lambda_A(0) > \lambda_A(y_o)$. Put

$$s_o = \frac{1}{2}[\mu_A(0) + \mu_A(x_o)] \Rightarrow s_o < \mu_A(x_o), 0 \leq \mu_A(0) < s_o < 1 \Rightarrow x_o \in U(\mu_A; s_o).$$

Since $U(\mu_A; s_o)$ is an H-ideal of X, we have $0 \in U(\mu_A; s_o) \Rightarrow \mu_A(0) \geq s_o$, which is contradiction. Therefore $\mu_A(0) \geq \mu_A(x)$, for all $x \in X$. Similarly by taking $t_o = \frac{1}{2}[\lambda_A(0) + \lambda_A(y_o)]$, we can show $\lambda_A(0) \leq \lambda_A(y)$, for any $y \in X$. If possible assume $x_o, y_o, z_o \in X$ such that $\mu_A(x_o * z_o) < min\{\mu_A(x_o * (y_o * z_o)), \mu_A(y_o)\}$.
Put $s_o = \frac{1}{2}[\mu_A(x_o * z_o) + min\{\mu_A(x_o * (y_o * z_o)), \mu_A(y_o)\}]$
$\Rightarrow s_o > \mu_A(x_o * z_o)$ and $s_o < min\{\mu_A(x_o * (y_o * z_o)), \mu_A(y_o)\}$
$\Rightarrow s_o > \mu_A(x_o * z_o), s_o < \mu_A(x_o * (y_o * z_o))$ and $s_o < \mu_A(y_o)$
$\Rightarrow x_o * z_o \notin U(\mu_A; s_O), x_o * (y_o * z_o) \in U(\mu_A; s_o)$ and $y_o \in U(\mu_A; s_o)$,
which is contradiction to H-ideal $U(\mu_A; s_o)$.
Therefore $\mu_A(x*z) \geq min\{\mu_A(x*(y*z)), \mu_A(y)\}$, for any $x, y, z \in X$. Similarly we can prove $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\}$, for any $x, y, z \in X$.
Hence $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy H-ideal of a BCK-algebra X.

Theorem 2.12 $A = (X, \mu_A, \lambda_A)$ *is an intuitionistic fuzzy closed H-ideal of a BCK-algebra X if and only if the non-empty upper s-level cut* $U(\mu_A; s)$ *and the non-empty lower t-level cut* $L(\lambda_A; t)$ *are closed H-ideal of* X*, for any* $s, t \in [0, 1]$.

Proof. Suppose $A = (X, \mu_A, \lambda_A)$ is an intuitionistic fuzzy closed H-ideal of a BCK-algebra X. We have $\mu_A(0 * x) \geq \mu_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$, for any $x \in X$.
For $x \in U(\mu_A; s) \Rightarrow x \in X$ and $\mu_A(x) \geq s \Rightarrow \mu_A(0*x) \geq s \Rightarrow 0*x \in U(\mu_A; s)$. And $x \in L(\lambda_A; t) \Rightarrow x \in X$ and $\lambda_A(x) \leq t \Rightarrow \lambda_A(0 * x) \leq t \Rightarrow 0 * x \in L(\lambda_A; t)$. Therefore $U(\mu_A; s)$ and $L(\lambda_A; t)$ are closed H-ideals of X.
Converse, it is enough to show that $\mu_A(0 * x) \geq \mu_A(x)$ and $\lambda_A(0 * x) \leq \lambda_A(x)$, for any $x \in X$. If possible, assume $x_o \in X$ such that $\mu_A(0 * x_o) < \mu_A(x_o)$.
Take $s_o = \frac{1}{2}[\mu_A(0*x)+\mu_A(x_o)] \Rightarrow \mu_A(0*x_o) < s_o < \mu_A(x_o) \Rightarrow x_o \in U(\mu_A; s_o)$, but $0 * x_o \notin U(\mu_A; s_o)$, which is contradiction to closed H-ideal.
Hence $\mu_A(0 * x) \geq \mu_A(x)$, for any $x \in X$. Similarly we can prove that $\lambda_A(0 * x) \leq \lambda_A(x)$, for any $x \in X$.

Corollary 2.13 *If $A = (X, \mu_A, \lambda_A)$ be an intuitionistic fuzzy closed H-ideal of X, then the sets $J = \{x \in X/\mu_A(x) = \mu_A(0)\}$ and $K = \{x \in X/\lambda_A(x) = \lambda_A(0)\}$ are H-ideal of X.*

Proof. Since $0 \in X, \mu_A(0) = \mu_A(0)$ and $\lambda_A(0) = \lambda_A(0)$ implies $0 \in J$ and $0 \in K$, So $J \neq \Phi$ and $K \neq \Phi$. Let $x * (y * z) \in J$ and $y \in J \Rightarrow \mu_A(x * (y * z)) = \mu_A(0)$ and $\mu_A(y) = \mu_A(0)$. Since $\mu_A(x*z) \geq min\{\mu_A(x*(y*z)), \mu_A(y)\} = \mu_A(0)$ $\Rightarrow \mu_A(x * z) \geq \mu_A(0)$, but $\mu_A(0) \geq \mu_A(x * z)$. It follows that $x * z \in J$, for all $x, y, z \in X$. Hence J is H-ideal of X. Similarly we can prove K is H-ideal of X.

Definition 2.14 *Let f be a mapping on a set X and $A = (X, \mu_A, \lambda_A)$ an intuitionistic fuzzy set in X. Then the fuzzy sets u and v on $f(X)$ defined by $u(y) = \sup_{x \in f^{-1}(y)} \mu_A(x)$ and $v(y) = \inf_{x \in f^{-1}(y)} \lambda_A(x)$, for all $y \in f(X)$, is called image of A under f. If u, v are fuzzy sets in $f(X)$ then the fuzzy sets $\mu_A = uof$ and $\lambda_A = vof$ is called the pre-image of u and v under f.*

Theorem 2.15 *Let $f : X \to X'$ be an onto homomorphism of BCK-algebras. If $A' = (X', u, v)$ is an intuitionistic fuzzy H-ideal of X', then the pre-image of A' under f is an intuitionistic fuzzy H-ideal of X.*

Proof. Let $A = (X, \mu_A, \lambda_A)$, where $\mu_A = uof$ and $\lambda_A = vof$ is the pre-image of $A' = (X', u, v)$ under f. Since $A' = (X', u, v)$ is an intuitionistic fuzzy H-ideal of X', we have $u(0') \geq u(f(x)) = \mu_A(x)$ and $v(0') \leq v(f(x)) = \lambda_A(x)$. On other hand $u(0') = u(f(0)) = \mu_A(0)$ and $v(0') = v(f(0)) = \lambda_A(0)$. Therefore $\mu_A(0) \geq \mu_A(x)$ and $\lambda_A(0) \leq \lambda_A(x)$, for all $x \in X$. Now we show that
(1). $\mu_A(x * z) \geq min\{\mu_A(x * (y * z)), \mu_A(y)\}$,
(2). $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\}$, for any $x, y, z \in X$.
We have
$\mu_A(x * z) = u(f(x * z)) = u(f(x) * f(z)) \geq min\{u(f(x) * (y' * f(z))), u(y')\}$, for $y' \in X'$. Since f is onto homomorphism, there is $y \in X$ such that $f(y) = y'$. Thus

$$\begin{aligned}
\mu_A(x * z) &\geq \min\{u(f(x) * (y' * f(z))), u(y')\} \\
&= \min\{u(f(x) * (f(y) * f(z))), u(f(y))\} \\
&= \min\{u(f(x * (y * z)), u(f(y))\} \\
&= \min\{\mu_A((x * (y * z)), \mu_A(y)\},
\end{aligned}$$

for all $x, y, z \in X$. Therefore, the result $\mu_A(x*z) \geq min\{\mu_A(x*(y*z)), \mu_A(y)\}$, is true for all $x, y, z \in X$, because y' is an arbitrary element of X' and f is onto mapping. Similarly, we can prove $\lambda_A(x * z) \leq max\{\lambda_A(x * (y * z)), \lambda_A(y)\}$, for any $x, y, z \in X$. Hence the pre-image $A = (X, \mu_A, \lambda_A)$, of A' is an intuitionistic H-ideal of X.

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